

Section : MATHEMATIQUES

Correction de la série

Analyse aux BAC

2^{ème} Partie

entre 1991 et 2008

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Problème 01: BAC 2008 - Session Principale

1) $g(x) = \frac{1}{x^2} + 1 - 4 \ln x$

a) $D_g =]0, +\infty[$ $g'(x) = -\frac{2}{x^3} - \frac{4}{x} < 0$

x	0	α	$+\infty$
g'		-	
g	$+\infty$		$-\infty$

$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} \left(\frac{1}{x^2} + 1 - 4 \ln x \right) = +\infty$
 $\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} \left(\frac{1}{x^2} + 1 - 4 \ln x \right) = -\infty$

b) g est strictement décroissante sur $]0, +\infty[\Rightarrow g$ réalise une bijection de $]0, +\infty[\rightarrow \mathbb{R}$
et comme $0 \in \mathbb{R} \Rightarrow \exists ! \alpha \in]0, +\infty[/ g(\alpha) = 0$

car $g(1) = 2 > 0$
 $g(2) = \frac{1}{4} + 1 - 4 \ln 2 < 0 \Rightarrow \boxed{1 < \alpha < 2}$

c) $\frac{x}{g(x)} \begin{array}{c|c} 0 & \alpha & +\infty \\ \hline & + & \phi & - \end{array}$

2) $f(x) = \frac{\ln x}{(1+x^2)^2}$ $D_f =]0, +\infty[$

a) $x > 0 \Rightarrow f$ est strictement croissante sur $]0, +\infty[$
 $1+x^2 > 0$

$f'(x) = \frac{\frac{1}{x}(1+x^2)^2 - \ln x \cdot 2(2x)(1+x^2)}{(1+x^2)^4}$

$\Rightarrow f'(x) = \frac{(1+x^2) \left[\frac{1}{x} + x - 4x \ln x \right]}{(1+x^2)^4}$

$\Rightarrow f'(x) = \frac{x \left[\frac{1}{x^2} + 1 - 4 \ln x \right]}{(1+x^2)^3} = \frac{x \cdot g(x)}{(1+x^2)^3}$

b) $\text{Sign}(f') = \text{Sign}(g(x))$

x	0	α	$+\infty$
$g(x)$		+	ϕ
$f(x)$	$-\infty$		0

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{(1+x^2)^2} = -\infty$
 $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{\ln x}{(1+x^2)^2} = 0$

3) a) $T: y = f'(1)(x-1) + f(1)$
car $f'(1) = \frac{1 \cdot g(1)}{(1+1)^3} = \frac{2}{8} = \frac{1}{4}$ et $f(1) = 0$
 $\Rightarrow T: \boxed{y = \frac{1}{4}x - \frac{1}{4}}$

b) $\forall x > 0, \ln x \leq x-1$
on pose $A(x) = \ln x - x + 1$ $D_A =]0, +\infty[$
 $A'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$

x	0	1	$+\infty$
A'		+	ϕ
A		0	-

$A(1) = 0$: maximum absolu
 $\Rightarrow \forall x > 0, A(x) \leq 0$
 $\Rightarrow \ln x - x + 1 \leq 0$
 $\Rightarrow \boxed{\ln x \leq x-1}$

• Dq $f(x) - \frac{1}{4}(x-1) \leq (x-1) \left[\frac{4 - (1+x^2)^2}{4(1+x^2)^2} \right]$

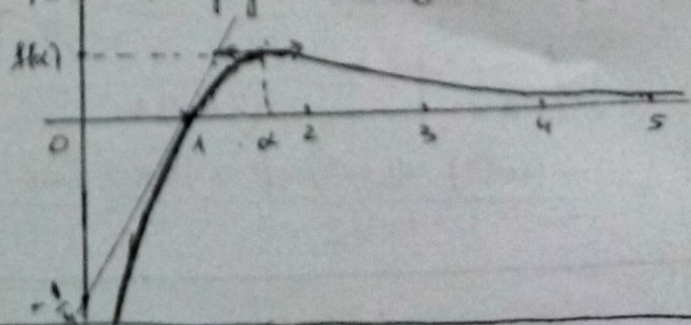
ma $\ln x \leq x-1$
 $\Rightarrow \frac{\ln x}{(1+x^2)^2} - \frac{1}{4}(x-1) \leq (x-1) \left[\frac{1}{(1+x^2)^2} - \frac{1}{4} \right]$
 $\Rightarrow \boxed{f(x) - \frac{1}{4}(x-1) \leq (x-1) \left[\frac{4 - (1+x^2)^2}{4(1+x^2)^2} \right]}$

c) ma $4 - (1+x^2)^2 = (2 - (1+x^2))(2 + (1+x^2))$
 $= (1-x^2)(3+x^2) = -(x-1)(x+1)(3+x^2)$
 $\Rightarrow f(x) - \frac{1}{4}(x-1) \leq -\frac{(x-1)^2(x+1)(3+x^2)}{4(1+x^2)^2} \leq 0$

$\Rightarrow f(x) \leq \frac{1}{4}(x-1) \Leftrightarrow f(x) \leq \frac{1}{4}x - \frac{1}{4}$

$\Rightarrow$ Egal toujours située au dessous de T

d) $x=0$: Asymptote verticale
 $y=0$: Asymptote horiz en $x \rightarrow +\infty$



$$\text{II} \quad F(x) = \int_{\frac{1}{x}}^x f(t) dt \quad / x > 0$$

1) $f(t) \in \text{sur }]0, +\infty[\rightarrow f$ admet au moins une primitive $\varphi \in \text{sur }]0, +\infty[/ \varphi'(t) = f(t)$

$$\Rightarrow F(x) = \left[\varphi(t) \right]_{\frac{1}{x}}^x = \varphi(x) - \varphi\left(\frac{1}{x}\right)$$

ou $\varphi \in \text{sur }]0, +\infty[$

$$\text{et } \forall x > 0, \frac{1}{x} > 0 \Rightarrow \left(\frac{1}{x}\right) \in]0, +\infty[= D_\varphi$$

$$\Rightarrow F \text{ est } \in \text{sur }]0, +\infty[$$

$$\forall x > 0, F'(x) = [\varphi(x)]' - [\varphi(\frac{1}{x})]'$$

$$\Rightarrow F'(x) = \varphi'(x) - \left(-\frac{1}{x^2}\right) \varphi'\left(\frac{1}{x}\right)$$

$$\Rightarrow F'(x) = f(x) + \frac{1}{x^2} \cdot f\left(\frac{1}{x}\right) = \frac{\ln x}{(1+x^2)^2} + \frac{1}{x^2} \cdot \frac{-\ln x}{(1+\frac{1}{x^2})^2}$$

$$\Rightarrow F'(x) = \frac{\ln x}{(1+x^2)^2} - \frac{x^2 \ln x}{(x^2+1)^2} = \frac{(1-x^2) \ln x}{(x^2+1)^2}$$

$$2) h(x) = \lg(x) \quad / x \in]0, \frac{\pi}{2}[$$

$$a) h \in \text{sur }]0, \frac{\pi}{2}[, h'(x) = 1 + \lg x > 0$$

$\Rightarrow h \in \text{strict } \nearrow \text{ sur }]0, \frac{\pi}{2}[\Rightarrow h$ réalise une bijection de $]0, \frac{\pi}{2}[\rightarrow]0, +\infty[$ car $\lim_{x \rightarrow \frac{\pi}{2}^-} h(x) = +\infty$

$$b) h \in \text{sur }]0, \frac{\pi}{2}[\Rightarrow h^{-1} \text{ est } \in \text{sur } \mathbb{R}^+ \text{ et } h'(x) \neq 0$$

$$(h^{-1})'(x) = \frac{1}{h'(y)} = \frac{1}{1 + \lg(y)}$$

$$a) h(y) = x \Leftrightarrow \lg(y) = x \Rightarrow (h^{-1})'(x) = \frac{1}{1+x^2}$$

$$c) \text{ Mg } F(x) = \frac{1}{2} \left(h^{-1}\left(\frac{1}{x}\right) - h^{-1}(x) \right) + \frac{x \ln x}{1+x^2}$$

$$\text{on pose } G(x) = \frac{1}{2} \left(h^{-1}\left(\frac{1}{x}\right) - h^{-1}(x) \right) + \frac{x \ln x}{1+x^2}$$

$G \in \text{sur }]0, +\infty[$ car $h^{-1} \in \text{sur } \mathbb{R}^+$

$$G'(x) = \frac{1}{2} \left(\left(-\frac{1}{x^2}\right) (h^{-1})'\left(\frac{1}{x}\right) - (h^{-1})'(x) \right) + \left(\frac{x \ln x}{1+x^2} \right)'$$

$$= \frac{1}{2} \left(-\frac{1}{x^2} \cdot \frac{1}{1+\frac{1}{x^2}} - \frac{1}{1+x^2} \right) + \frac{(\ln x + 1)(1+x^2) - 2x^2 \ln x}{(1+x^2)^2}$$

$$= \frac{1}{2} \left(\frac{-2}{1+x^2} \right) + \frac{\ln x + x^2 \ln x + 1 + x^2 - 2x^2 \ln x}{(1+x^2)^2}$$

$$= \frac{-(1+x^2) + (1+x^2) + (1-x^2) \ln x}{(1+x^2)^2} = \frac{(1-x^2) \ln x}{(1+x^2)^2}$$

$$\text{donc } G'(x) = F'(x) \Rightarrow G(x) = F(x) + k$$

$$\text{or } F(1) = \int_1^1 f(t) dt = 0 \Rightarrow k = 0$$

$$\text{et } G(1) = \frac{1}{2} \left(h^{-1}\left(\frac{1}{1}\right) - h^{-1}(1) \right) + \frac{1 \ln 1}{1+1} = 0$$

$$\text{donc } F(x) = G(x) \Leftrightarrow F(x) = \frac{1}{2} \left(h^{-1}\left(\frac{1}{x}\right) - h^{-1}(x) \right) + \frac{x \ln x}{1+x^2}$$

$$3) a) \lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow 0^+} \frac{1}{2} \left(h^{-1}\left(\frac{1}{x}\right) - h^{-1}(x) \right) + \frac{x \ln x}{1+x^2}$$

$$\text{or } \lim_{x \rightarrow 0^+} h^{-1}\left(\frac{1}{x}\right) = \lim_{t \rightarrow +\infty} h^{-1}(t) = \frac{\pi}{2}$$

$$\text{et } \lim_{x \rightarrow 0^+} h^{-1}(x) = 0 \quad \text{et } \lim_{x \rightarrow 0^+} \frac{x \ln x}{1+x^2} = 0$$

$$\Rightarrow \lim_{x \rightarrow 0^+} F(x) = \frac{\pi}{4}$$

$$\bullet \lim_{x \rightarrow +\infty} F(x) = \lim_{x \rightarrow +\infty} \frac{1}{2} \left(h^{-1}\left(\frac{1}{x}\right) - h^{-1}(x) \right) + \frac{x \ln x}{1+x^2}$$

$$\text{or } \lim_{x \rightarrow +\infty} h^{-1}\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0^+} h^{-1}(t) = 0$$

$$\text{et } \lim_{x \rightarrow +\infty} h^{-1}(x) = \frac{\pi}{2}$$

$$\text{et } \lim_{x \rightarrow +\infty} \frac{x \ln x}{1+x^2} = \lim_{x \rightarrow +\infty} \frac{\frac{\ln x}{x}}{\frac{1}{x^2} + 1} = 0$$

$$\lim_{x \rightarrow +\infty} F(x) = -\frac{\pi}{4}$$

$$b) F'(x) = \frac{(1-x^2) \ln x}{(x^2+1)^2}$$

F' s'annule en 1.

$$c) G(x) = F(x) \text{ si } x > 0$$

$$G(0) = \frac{\pi}{4}$$

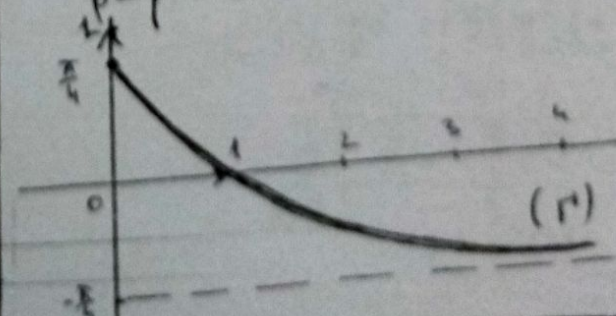
$$\text{Comme } \lim_{x \rightarrow 0^+} F(x) = \frac{\pi}{4} = G(0)$$

$\Rightarrow G$ est le prolongement de F par continuité en 0 alors

$$\forall x > 0, G'(x) = F'(x)$$

$$\lim_{x \rightarrow +\infty} G(x) = -\frac{\pi}{4} \Rightarrow \gamma = -\frac{\pi}{4} \text{ A l'horiz en } +\infty$$

Rq: La $\in$ de G en 0 et ne peut pas être étudiée avec les données proposées



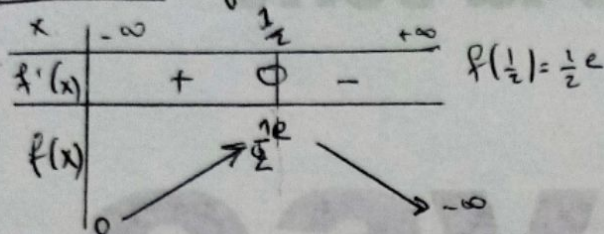
Problème 02: BAC 2008 - Session de Contrôle

$$f(x) = (1-x) \cdot e^{2x} \quad D_f = \mathbb{R}$$

$$1) f \in \mathcal{C}^\infty \text{ sur } \mathbb{R}, f'(x) = -e^{2x} + 2(1-x)e^{2x}$$

$$\Leftrightarrow f'(x) = (1-2x) \cdot e^{2x}$$

$$\text{on résoud : } f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$$



$$\cdot \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} (1-x)e^{2x} = \lim_{x \rightarrow -\infty} e^{-\frac{1}{2}} \left(\frac{1}{2} \right)^{2x} = 0$$

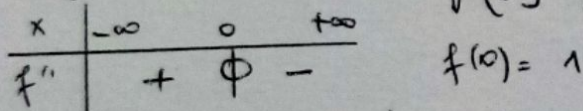
$$\Rightarrow y = 0 \text{ Asymptote horizontale en } -\infty$$

$$\cdot \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} (1-x)e^{2x} = -\infty$$

$$2) a) \text{ Point d'inflexion : } f'(x) = (1-2x)e^{2x}$$

$$f' \in \mathcal{C}^\infty \text{ sur } \mathbb{R}, f''(x) = -2e^{2x} + 2(1-2x)e^{2x}$$

$$\Leftrightarrow f''(x) = -4xe^{2x} \Rightarrow \text{sgn}(f'') = \text{sgn}(-4x)$$



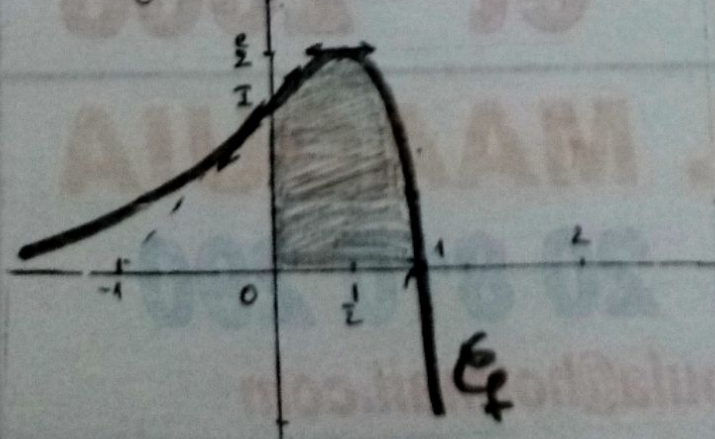
f'' s'annule en 0 et change de signe.
 $\Rightarrow I(0,1)$ est un point d'inflexion pm $\mathcal{C}_f$

$$b) T_I: y = f'(0)(x-0) + f(0)$$

$$\cdot \begin{cases} f'(0) = 1 \\ f(0) = 1 \end{cases} \Rightarrow T_I: y = x + 1$$

$$c) \text{ Branche inférieure : } \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \left(\frac{1-x}{x} \right) e^{2x} = -\infty$$

$$\Rightarrow \mathcal{C}_f \text{ admet une BI en } (+\infty) \text{ de direction } (0, -1)$$



$$3) A = \int_0^1 f(x) dx = \int_0^1 (1-x)e^{2x} dx$$

$$\text{on pose } \begin{cases} u(x) = 1-x \rightarrow u'(x) = -1 \\ v'(x) = e^{2x} \rightarrow v(x) = \frac{1}{2}e^{2x} \end{cases}$$

$$\Rightarrow A = \left[\frac{1}{2}(1-x)e^{2x} \right]_0^1 + \frac{1}{2} \int_0^1 e^{2x} dx$$

$$\Leftrightarrow A = \left[\frac{1}{2}(1-x)e^{2x} + \frac{1}{4}e^{2x} \right]_0^1 = \left[\left(\frac{3}{4} - \frac{1}{2}x \right) e^{2x} \right]_0^1$$

$$= \frac{1}{4}e^2 - \frac{3}{4} \Rightarrow \boxed{A = \frac{e^2 - 3}{4}} \quad (4A)$$

$$4) \forall n \in \mathbb{N}^*, I_n = \int_0^1 (1-x)^n e^{2x} dx$$

$$a) 0 \leq x \leq 1 \Leftrightarrow 0 \leq 2x \leq 2$$

$$e^1 \Rightarrow 1 \leq e^{2x} \leq e^2$$

$$\text{or } 0 \leq 1-x \leq 1 \Rightarrow (1-x)^n \geq 0$$

$$\text{d'où } (1-x)^n \leq (1-x)^n e^{2x} \leq (1-x)^n e^2$$

$$\text{or } \begin{cases} x \mapsto (1-x)^n \in \mathcal{M}[0,1] \\ x \mapsto (1-x)^n e^{2x} \in \mathcal{M}[0,1] \\ x \mapsto (1-x)^n e^2 \in \mathcal{M}[0,1] \end{cases} \Rightarrow$$

$$\int_0^1 (1-x)^n dx \leq \int_0^1 (1-x)^n e^{2x} dx \leq e^2 \int_0^1 (1-x)^n dx$$

$$\Rightarrow \left[\frac{-1}{n+1} (1-x)^{n+1} \right]_0^1 \leq I_n \leq \left[\frac{-e^2}{n+1} (1-x)^{n+1} \right]_0^1$$

$$\Rightarrow 0 + \frac{1}{n+1} \leq I_n \leq 0 + \frac{e^2}{n+1}$$

$$\Rightarrow \forall n \in \mathbb{N}^* \quad \boxed{\frac{1}{n+1} \leq I_n \leq \frac{e^2}{n+1}}$$

$$\cdot \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \quad \Rightarrow \quad \boxed{\lim I_n = 0}$$

$$\cdot \lim_{n \rightarrow \infty} \frac{e^2}{n+1} = 0$$

$$b) 2I_{n+1} = \int_0^1 2(1-x)^{n+1} e^{2x} dx$$

$$\text{on pose } \begin{cases} u(x) = (1-x)^{n+1} \rightarrow u'(x) = -(n+1)(1-x)^n \\ v'(x) = 2e^{2x} \rightarrow v(x) = e^{2x} \end{cases}$$

$$\Rightarrow 2I_{n+1} = \left[(1-x)^{n+1} e^{2x} \right]_0^1 + (n+1) \int_0^1 (1-x)^n e^{2x} dx$$

$$= 0 - 1 + (n+1) I_n$$

$$\Rightarrow \forall n \in \mathbb{N}^* \quad \boxed{2I_{n+1} = (n+1) I_n - 1}$$

$$\Leftrightarrow 2I_{n+1} = nI_n + I_n - 1$$

$$\Rightarrow n I_n = 2 I_{n+1} - I_n + 1$$

$$\text{or } \lim_{n \rightarrow \infty} I_n = 0 \Rightarrow \lim_{n \rightarrow \infty} I_{n+1} = 0$$

$$\text{alors } \boxed{\lim_{n \rightarrow \infty} (n I_n) = 1}$$

$$5) \forall n \in \mathbb{N}^*, U_n = \frac{2^n}{n!} I_n$$

$$a) \text{ On } \forall n \in \mathbb{N}^*, \frac{2^{n-1}}{n!} \leq 1$$

$$\text{pm } n=1, \frac{2^0}{1!} = 1 \leq 1 \text{ vérifié}$$

$$\text{Supposons que } \frac{2^{n-1}}{n!} \leq 1 \text{ vrai}$$

$$\text{Démontrons que } \frac{2^n}{(n+1)!} \leq 1$$

$$\text{en effet : } \frac{2^n}{(n+1)!} = \frac{2 \cdot 2^{n-1}}{(n+1) \cdot n!}$$

$$\text{or } n \geq 1 \Rightarrow n+1 \geq 2 \Rightarrow \frac{1}{n+1} \leq \frac{1}{2}$$

$$\text{alors } \frac{2}{n+1} \leq 1 \Rightarrow \frac{2^n}{(n+1)!} \leq 1$$

$$\text{or } \frac{2^{n-1}}{n!} \leq 1 \Rightarrow \frac{2^n}{(n+1)!} \leq 1$$

$$\text{Concl: } \forall n \in \mathbb{N}^* \quad \boxed{\frac{2^{n-1}}{n!} \leq 1}$$

$$b) \text{ On } U_n \leq \frac{2e^2}{n+1}$$

$$\text{or } I_n \leq \frac{e^2}{n+1} \text{ d'après (a)}$$

$$\text{et } \frac{2^{n-1}}{n!} \leq 1$$

$$\text{Puissant: } \frac{2^{n-1}}{n!} I_n \leq \frac{e^2}{n+1}$$

$$\text{d'où } \frac{2^n}{n!} I_n \leq \frac{2e^2}{n+1} \Rightarrow \boxed{U_n \leq \frac{2e^2}{n+1}}$$

$$c) \text{ d'autre part : } \left. \begin{array}{l} I_n \geq 0 \\ \frac{2^n}{n!} > 0 \end{array} \right\} \Rightarrow U_n \geq 0$$

$$\text{alors } 0 \leq U_n \leq \frac{2e^2}{n+1}$$

$$\text{Comme } \lim_{n \rightarrow \infty} \frac{2e^2}{n+1} = 0 \Rightarrow \boxed{\lim_{n \rightarrow \infty} U_n = 0}$$

$$6) a) \text{ On } \forall n \in \mathbb{N}^* \quad U_{n+1} = \frac{2^n}{(n+1)!} + U_n$$

$$\text{or } U_{n+1} = \frac{2^{n+1}}{(n+1)!} I_{n+1} = \frac{2^n}{(n+1)!} (2I_{n+1})$$

$$\Rightarrow U_{n+1} = \frac{2^n}{(n+1)!} [(n+1) I_n - 1]$$

$$\Rightarrow U_{n+1} = \frac{2^n (n+1)}{(n+1)!} I_n - \frac{2^n}{(n+1)!}$$

$$\Rightarrow \boxed{U_{n+1} = U_n - \frac{2^n}{(n+1)!}} \quad \forall n \in \mathbb{N}^*$$

$$b) \text{ On } \forall n \in \mathbb{N}^* \quad U_n = \frac{1}{2} \left[e^2 - \sum_{k=0}^n \frac{2^k}{k!} \right]$$

$$\text{On } n=1, U_1 = \frac{2^1}{1!} I_1 = 2 I_1$$

$$\text{or } I_1 = \int_0^1 (1-x) e^{2x} dx = A = \frac{e^2 - 3}{4}$$

$$\Rightarrow U_1 = 2 \left(\frac{e^2 - 3}{4} \right) = \frac{1}{2} [e^2 - 3] = \frac{1}{2} \left[e^2 - \left(\frac{2^0}{0!} + \frac{2^1}{1!} \right) \right] = \frac{1}{2} \left[e^2 - \sum_{k=0}^1 \frac{2^k}{k!} \right]$$

$$\Rightarrow U_1 \text{ vérifié}$$

$$\text{Supposons que } U_n = \frac{1}{2} \left[e^2 - \sum_{k=0}^n \frac{2^k}{k!} \right] \text{ vrai}$$

$$\text{Démontrons que } U_{n+1} = \frac{1}{2} \left[e^2 - \sum_{k=0}^{n+1} \frac{2^k}{k!} \right]$$

$$\text{en effet : } U_{n+1} = U_n - \frac{2^n}{(n+1)!} = \frac{1}{2} \left[e^2 - \sum_{k=0}^n \frac{2^k}{k!} \right] - \frac{2^n}{2(n+1)!}$$

$$\Rightarrow U_{n+1} = \frac{1}{2} \left[e^2 - \sum_{k=0}^n \frac{2^k}{k!} - \frac{2^{n+1}}{(n+1)!} \right] = \frac{1}{2} \left[e^2 - \sum_{k=0}^{n+1} \frac{2^k}{k!} \right]$$

$$\text{Concl: } \forall n \in \mathbb{N}^* \quad \boxed{U_n = \frac{1}{2} \left[e^2 - \sum_{k=0}^n \frac{2^k}{k!} \right]}$$

$$c) \text{ on pose } V_n = \sum_{k=0}^n \frac{2^k}{k!} \Rightarrow U_n = \frac{1}{2} [e^2 - V_n]$$

$$\Rightarrow 2U_n = e^2 - V_n \Rightarrow V_n = e^2 - 2U_n$$

$$\text{or } \lim_{n \rightarrow \infty} U_n = 0 \Rightarrow \boxed{\lim_{n \rightarrow \infty} V_n = e^2}$$

$$\text{concl: } V_n \text{ converge vers } e^2$$

Exercice 03: BAC 2007 - Session de Contrôle

$$f(x) = e^{-\sqrt{x-1}} \quad \text{Def} = [1, +\infty[$$

$$1) a) \text{ } \frac{1}{x-1} \text{ de fen } 1^+ : f(1) = e^0 = 1$$

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \rightarrow 1^+} \frac{e^{-\sqrt{x-1}} - 1}{x-1}$$

$$\text{on pose } X = \sqrt{x-1}, \text{ si } x \rightarrow 1^+ \Rightarrow X \rightarrow 0^+$$

$$l = \lim_{X \rightarrow 0^+} \frac{e^{-X} - 1}{X^2} = \lim_{X \rightarrow 0^+} \frac{-e^{-X}}{2X} = \lim_{X \rightarrow 0^+} \frac{-e^{-X}}{2X}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{e^x - 1}{x} \right) \cdot \frac{1}{xe^x} = -\infty \text{ car } \begin{cases} \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x} = 1 \\ \lim_{x \rightarrow 0^+} xe^x = 0^+ \end{cases}$$

$\Rightarrow f$ n'est pas $\leq$ en 1 et E_f admet en $(1, 1)$ une demi-tangente verticale dirigée vers bas

b) $f \in \text{som }]1, +\infty[$ car $x-1 \geq 0$

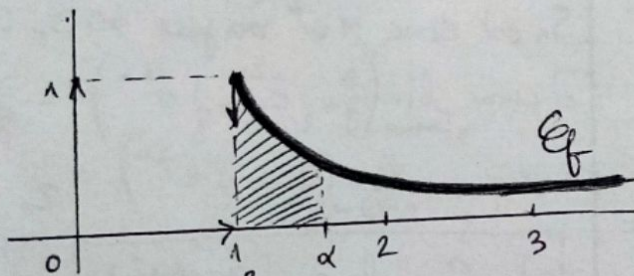
et $f \in \text{som }]1, +\infty[$

$$f'(x) = -\frac{1}{2\sqrt{x-1}} e^{-\sqrt{x-1}} < 0$$

x	1	$+\infty$
$f(x)$	1	—
$f(x)$	1	0

$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} e^{-\sqrt{x-1}} = 0$
car $\lim_{x \rightarrow +\infty} -\sqrt{x-1} = -\infty$

$\Rightarrow y=0$ Asymptote horizontale en $+\infty$



2) $F(x) = \int_1^{1+(\ln x)^2} f(t) dt$ $D_f =]0, 1[$

a) car $f \in \text{som }]1, +\infty[\Rightarrow f$ admet une primitive $H \in \text{som }]1, +\infty[$ / $H'(t) = f(t)$

$$\text{donc } F(x) = [H(t)]_1^{1+(\ln x)^2} = H(1+(\ln x)^2) - H(1)$$

or $x \mapsto 1+(\ln x)^2 \in \text{som }]0, 1[\Rightarrow$
 $\text{et } [1+(\ln x)^2] \subset]1, +\infty[= D_f \Rightarrow$

$$F \in \text{som }]0, 1[$$

$$\begin{aligned} F'(x) &= [H(1+(\ln x)^2)]' - [H(1)]' \\ &= 2\left(\frac{1}{x}\right) \ln x \cdot H'(1+(\ln x)^2) - 0 \\ &= \frac{2}{x} \ln x \cdot f(1+(\ln x)^2) = \frac{2}{x} \ln x \cdot e^{-\sqrt{(\ln x)^2}} \\ &= \frac{2}{x} \ln x \cdot e^{-|\ln x|} = \frac{2}{x} \ln x \cdot e^{\ln x} = \frac{2}{x} \ln x \cdot x \end{aligned}$$

$$\Rightarrow F'(x) = 2 \ln x \text{ car } x \in]0, 1[\Rightarrow \ln x \leq 0$$

b) $F(x) = 2(x \ln x - x) + k$

$$\Rightarrow F(x) = 2x \ln x - 2x + k$$

$$\text{or } F(1) = \int_1^1 f(t) dt = 0 \text{ et } F(1) = k - 2$$

$$\Rightarrow k = 2$$

$$\text{donc : } \forall x \in]0, 1[, F(x) = 2x \ln x - 2x + 2$$

3) $\alpha \geq 1$ $S(\alpha)$: l'aire limitée par la courbe E_f et les droites: $y=0$, $x=1$, $x=\alpha$

a) $S(\alpha) = \int_1^\alpha f(x) dx$

$$\text{or } F(f(x)) = \int_1^\alpha \frac{1 + \ln(f(x))}{f(x)} dx$$

$$\begin{aligned} \text{et } 1 + \ln(f(x)) &= 1 + \ln(e^{-\sqrt{x-1}}) = 1 - \sqrt{x-1} \\ &= 1 + \alpha - x = \alpha \Rightarrow F(f(x)) = \int_1^\alpha f(x) dx \end{aligned}$$

$$\text{donc : } S(\alpha) = F(f(\alpha))$$

b) $S(\alpha) = F(f(\alpha)) = 2f(\alpha) \cdot \ln(f(\alpha)) - 2f(\alpha) + 2$
 $= 2e^{-\sqrt{\alpha-1}} \cdot \ln(e^{-\sqrt{\alpha-1}}) - 2e^{-\sqrt{\alpha-1}} + 2$

$$\Rightarrow S(\alpha) = -2\sqrt{\alpha-1} e^{-\sqrt{\alpha-1}} - 2e^{-\sqrt{\alpha-1}} + 2$$

$$\Rightarrow S(\alpha) = 2 - 2e^{-\sqrt{\alpha-1}} (\sqrt{\alpha-1} + 1) \text{ car}$$

$$\begin{aligned} \cdot \lim_{\alpha \rightarrow +\infty} e^{-\sqrt{\alpha-1}} (\sqrt{\alpha-1} + 1) &= \lim_{\alpha \rightarrow +\infty} \left(\frac{\sqrt{\alpha-1}}{e^{\sqrt{\alpha-1}}} + \frac{1}{e^{\sqrt{\alpha-1}}} \right) \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\frac{e^x}{x}} - \frac{1}{e^x} = 0 \text{ avec } x = \sqrt{\alpha-1} \end{aligned}$$

$$\Rightarrow \lim_{\alpha \rightarrow +\infty} S(\alpha) = 2$$

Problème 04 : BAC 2006 - Session Principale

$$f(x) = x + \ln(1 + e^{2x}) \quad D_f =]0, +\infty[$$

(A) 1) a) $f \in \text{som }]0, +\infty[$ car $1 + e^{2x} > 0$

$$\begin{aligned} f'(x) &= 1 + \frac{-2e^{2x}}{1 + e^{2x}} = \frac{1 + e^{2x} - 2e^{2x}}{1 + e^{2x}} \\ &= \frac{1 - e^{2x}}{1 + e^{2x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \geq 0 \quad \forall x \geq 0 \end{aligned}$$

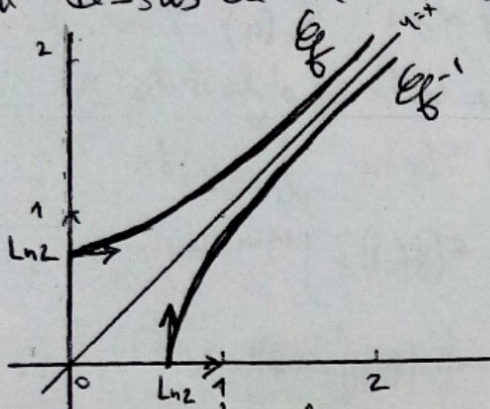
x	0	$+\infty$
$f'(x)$	0	+
$f(x)$	$\ln 2$	$+\infty$

$f(0) = \ln(2)$
 $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} x + \ln(1 + e^{2x}) = +\infty$
car $\lim_{x \rightarrow +\infty} e^{2x} = +\infty$

b) $\lim_{t \rightarrow +\infty} (f(t) - x) = \lim_{t \rightarrow +\infty} \ln(1 + e^{-2x}) = \ln 1 = 0$
 $\Rightarrow$ la droite $D: y = x$ est une asymptote oblique à $\mathcal{C}_f$ au voisinage de $(+\infty)$.

c) $f(x) - y = \ln(1 + e^{-2x}) > 0$

car $1 + e^{-2x} > 1$
 $\Rightarrow$ la courbe $\mathcal{C}_f$ est toujours située au dessus de la droite $D: y = x$



2) a) f continue et strictement $\nearrow$ sur $[0, +\infty[$
 $\Rightarrow f$ réalise une bijection de $[0, +\infty[$ sur $f([0, +\infty[) = [\ln 2, +\infty[$

b) $\mathcal{C}_f^{-1} = S_D(\mathcal{C}_f) / D: y = x$

3) a) $(1-t) - \frac{1}{1+t} = \frac{(1-t)(1+t)-1}{1+t} = \frac{-t^2}{1+t} \leq 0$
 $\Rightarrow 1-t \leq \frac{1}{1+t} \quad (1)$

et $\frac{1}{1+t} - 1 = \frac{-t}{1+t} \leq 0 \quad \forall t \geq 0 \quad (2)$

(1) et (2) $\Rightarrow 1-t \leq \frac{1}{1+t} \leq 1 \quad \forall t \geq 0$

b) pour $0 \leq x \Rightarrow \int_0^x (1-t) dt \leq \int_0^x \frac{1}{1+t} dt \leq \int_0^x 1 dt$

$\Rightarrow [t - \frac{1}{2}t^2]_0^x \leq [\ln|1+t|]_0^x \leq [t]_0^x$

$\Rightarrow x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x \quad \forall x \geq 0$

on pose $x = e^{-2t} \geq 0$

$\Rightarrow e^{-2t} - \frac{1}{2}(e^{-2t})^2 \leq \ln(1 + e^{-2t}) \leq e^{-2t}$

$\Rightarrow \forall t \geq 0, e^{-2t} - \frac{1}{2}e^{-4t} \leq \ln(1 + e^{-2t}) \leq e^{-2t}$

4) a) $S_n = \int_0^n (f(x) - x) dx = \int_0^n \ln(1 + e^{-2x}) dx$

ou $e^{-2t} - \frac{1}{2}e^{-4t} \leq \ln(1 + e^{-2t}) \leq e^{-2t}$

car $0 < n \Rightarrow \int_0^n (e^{-2t} - \frac{1}{2}e^{-4t}) dt \leq \int_0^n \ln(1 + e^{-2t}) dt \leq \int_0^n e^{-2t} dt$

$\Rightarrow [-\frac{1}{2}e^{-2t} + \frac{1}{8}e^{-4t}]_0^n \leq S_n \leq [-\frac{1}{2}e^{-2t}]_0^n$

$\Rightarrow -\frac{1}{2}e^{-2n} + \frac{1}{8}e^{-4n} - (-\frac{1}{2} + \frac{1}{8}) \leq S_n \leq -\frac{1}{2}e^{-2n} + \frac{1}{2}$

$\Rightarrow \boxed{\frac{3}{8} - \frac{1}{2}e^{-2n} + \frac{1}{8}e^{-4n} \leq S_n \leq \frac{1}{2} - \frac{1}{2}e^{-2n}}$

b) Montrons d'abord que $S_n \nearrow$

$S_{n+1} - S_n = \int_0^{n+1} \ln(1 + e^{-2t}) dt - \int_0^n \ln(1 + e^{-2t}) dt$
 $= \int_n^{n+1} \ln(1 + e^{-2t}) dt$

or $\ln(1 + e^{-2t}) \geq 0 \Rightarrow S_{n+1} - S_n \geq 0$
 et $n < n+1$ donc $\boxed{S_n \nearrow}$

et $S_n \leq \frac{1}{2} - \frac{1}{2}e^{-2n} \leq \frac{1}{2}$

$\Rightarrow S_n$ majorée par $\frac{1}{2}$.

S_n est donc $\nearrow$ et majorée $\Rightarrow S_n$ converge

et comme $\lim_{n \rightarrow +\infty} (\frac{3}{8} - \frac{1}{2}e^{-2n} + \frac{1}{8}e^{-4n}) = \frac{3}{8}$
 et que $\lim_{n \rightarrow +\infty} (\frac{1}{2} - \frac{1}{2}e^{-2n}) = \frac{1}{2}$ $\Rightarrow$

$\boxed{\lim S_n = l \in [\frac{3}{8}, \frac{1}{2}]}$

B) $U_0 = \int_0^{\ln 2} 1 dx$ $U_n = \int_0^{\ln 2} [f'(x)]^n dx$

1) $U_0 = [x]_0^{\ln 2} = \ln 2$

$U_1 = \int_0^{\ln 2} f'(x) dx = [f(x)]_0^{\ln 2} = f(\ln 2) - f(0)$
 $= \ln 2 + \ln(1 + e^{-2\ln 2}) - \ln 2 = \ln(1 + \frac{1}{4})$

$\Rightarrow \boxed{U_1 = \ln(\frac{5}{4})}$

2) a) or $f'(x) = \frac{e^{2x}-1}{e^{2x}+1} = 1 - \frac{2}{e^{2x}+1}$

or $0 \leq x \leq \ln 2 \Rightarrow 0 \leq 2x \leq \ln 4$

$\Rightarrow 1 \leq e^{2x} \leq 4 \Rightarrow 2 \leq e^{2x} + 1 \leq 5$

$\Rightarrow \frac{1}{5} \leq \frac{1}{e^{2x}+1} \leq \frac{1}{2} \Rightarrow -1 \leq -\frac{2}{e^{2x}+1} \leq -\frac{2}{5}$

$\Rightarrow 0 \leq 1 - \frac{2}{e^{2x}+1} \leq \frac{3}{5} \Rightarrow \boxed{0 \leq f'(x) \leq \frac{3}{5}}$

Concl: $\forall x \in [0, \ln 2]; \quad 0 \leq f'(x) \leq \frac{3}{5}$

b) d'où $0 \leq [f'(x)]^n \leq \left(\frac{3}{5}\right)^n$

alors $0 \leq \int_0^{\ln 2} [f'(x)]^n dx \leq \int_0^{\ln 2} \left(\frac{3}{5}\right)^n dx$

$\Rightarrow 0 \leq U_n \leq \left(\frac{3}{5}\right)^n \left(x\right)_0^{\ln 2}$

$\Rightarrow 0 \leq U_n \leq \left(\frac{3}{5}\right)^n \cdot \ln 2$

c) $\frac{3}{5} \in]-1, 1[\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{3}{5}\right)^n = 0 \Rightarrow \lim_{n \rightarrow \infty} U_n = 0$

3) a) $f''(x) = \left(\frac{e^{2x}-1}{e^{2x}+1}\right)' = \frac{2e^{2x}(e^{2x}+1) - 2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2}$
 $= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x}+1)^2} = \frac{4e^{2x}}{(e^{2x}+1)^2}$

$\Rightarrow 1 - f''(x) = 1 - \frac{4e^{2x}}{(e^{2x}+1)^2} = \frac{(e^{2x}+1)^2 - 4e^{2x}}{(e^{2x}+1)^2}$
 $= \frac{(e^{2x}-1)^2}{(e^{2x}+1)^2} = \left(\frac{e^{2x}-1}{e^{2x}+1}\right)^2 = [f'(x)]^2$

Concl: $1 - f''(x) = [f'(x)]^2$

b) $U_n = \int_0^{\ln 2} [f'(x)]^n dx = \int_0^{\ln 2} [f'(x)]^{n-2} \cdot [f'(x)]^2 dx$

$= \int_0^{\ln 2} [f'(x)]^{n-2} (1 - f''(x)) dx$

$\Rightarrow U_n = \int_0^{\ln 2} [f'(x)]^{n-2} dx - \int_0^{\ln 2} f''(x) \cdot [f'(x)]^{n-2} dx$

$= U_{n-2} - \left[\frac{1}{n-1} [f'(x)]^{n-1} \right]_0^{\ln 2}$

$= U_{n-2} - \frac{1}{n-1} \left([f'(\ln 2)]^{n-1} - [f'(0)]^{n-1} \right)$

$\Rightarrow U_n = U_{n-2} - \frac{1}{n-1} \left(\frac{3}{5}\right)^{n-1} \quad \forall n \geq 2$

c) Or Par récurrence que: $U_{2n} = U_0 - \sum_{k=1}^n \left(\frac{1}{2k-1}\right) \left(\frac{3}{5}\right)^{2k-1}$

pour $n=1$ $U_2 = U_0 - \frac{1}{1} \left(\frac{3}{5}\right)^1 = \text{vrai}$

Suppos $U_{2n} = U_0 - \sum_{k=1}^n \left(\frac{1}{2k-1}\right) \left(\frac{3}{5}\right)^{2k-1}$ vrai

Or $U_{2n+2} = U_0 - \sum_{k=1}^{n+1} \left(\frac{1}{2k-1}\right) \left(\frac{3}{5}\right)^{2k-1}$

en eff: $U_{2n+2} = U_{2n} - \frac{1}{2n+1} \left(\frac{3}{5}\right)^{2n+1}$
 $= U_0 - \sum_{k=1}^n \frac{1}{2k-1} \left(\frac{3}{5}\right)^{2k-1} - \frac{1}{2n+1} \left(\frac{3}{5}\right)^{2n+1}$

$\Rightarrow U_{2n+2} = U_0 - \sum_{k=1}^{n+1} \left(\frac{1}{2k-1}\right) \left(\frac{3}{5}\right)^{2k-1}$

Concl: $\forall n \in \mathbb{N}^*$ $U_{2n} = U_0 - \sum_{k=1}^n \left(\frac{1}{2k-1}\right) \left(\frac{3}{5}\right)^{2k-1}$

Or $U_{2n+1} = U_1 - \sum_{k=1}^n \frac{1}{2k} \left(\frac{3}{5}\right)^{2k}$

pour $n=1$ $U_3 = U_1 - \frac{1}{2} \left(\frac{3}{5}\right)^2 = \text{vrai}$

Suppos que: $U_{2n+1} = U_1 - \sum_{k=1}^n \frac{1}{2k} \left(\frac{3}{5}\right)^{2k}$

Or $U_{2n+3} = U_1 - \sum_{k=1}^{n+1} \frac{1}{2k} \left(\frac{3}{5}\right)^{2k}$

en eff: $U_{2n+3} = U_{2n+1} - \frac{1}{2n+2} \left(\frac{3}{5}\right)^{2n+2}$

$\Rightarrow U_{2n+3} = U_1 - \sum_{k=1}^n \frac{1}{2k} \left(\frac{3}{5}\right)^{2k} - \frac{1}{2(n+1)} \left(\frac{3}{5}\right)^{2(n+1)}$

$\Rightarrow U_{2n+3} = U_1 - \sum_{k=1}^{n+1} \left(\frac{1}{2k}\right) \left(\frac{3}{5}\right)^{2k}$

Concl: $\forall n \in \mathbb{N}^*$ $U_{2n+1} = U_1 - \sum_{k=1}^n \frac{1}{2k} \left(\frac{3}{5}\right)^{2k}$

4) $V_n = \sum_{k=1}^{2n} \frac{1}{k} \left(\frac{3}{5}\right)^k$

or: $U_{2n} + U_{2n+1} = U_0 + U_1 - \sum_{k=1}^n \frac{1}{2k-1} \left(\frac{3}{5}\right)^{2k-1} - \sum_{k=1}^n \frac{1}{2k} \left(\frac{3}{5}\right)^{2k}$

$\Rightarrow U_{2n} + U_{2n+1} = U_0 + U_1 - \sum_{k=1}^{2n} \frac{1}{k} \left(\frac{3}{5}\right)^k = U_0 + U_1 - V_n$

$\Rightarrow V_n = U_0 + U_1 - U_{2n} - U_{2n+1}$

or $\lim_{n \rightarrow \infty} U_n = 0 \Rightarrow \lim_{n \rightarrow \infty} U_{2n} = \lim_{n \rightarrow \infty} U_{2n+1} = 0$

$\Rightarrow \lim_{n \rightarrow \infty} V_n = U_0 + U_1 = \ln 2 + \ln\left(\frac{5}{2}\right) = \ln\left(\frac{5}{2}\right)$

Concl: V_n converge vers $\ln\left(\frac{5}{2}\right)$

Problème 05: BAC 2005 - Session Principale

$f(x) = \frac{e^{2x}}{1+e^x} \quad D_f = \mathbb{R}$

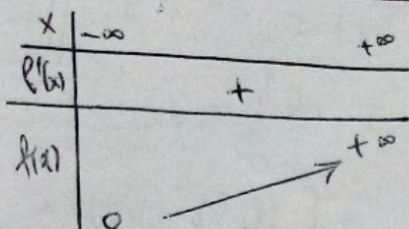
A) 1) $f \in \mathcal{C}^1$ sur $\mathbb{R}$ car $1+e^x \neq 0$

$f'(x) = \frac{2e^{2x}(1+e^x) - e^x(e^{2x})}{(1+e^x)^2}$

$\Rightarrow f'(x) = \frac{2e^{2x} + e^{3x}}{(1+e^x)^2} > 0 \quad \forall x \in \mathbb{R}$

$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{e^{2x}}{1+e^x} = \frac{0}{1} = 0$

$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{e^{2x}}{1+e^x} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \rightarrow +\infty} \frac{e^x}{e^{-x} + 1} = +\infty$

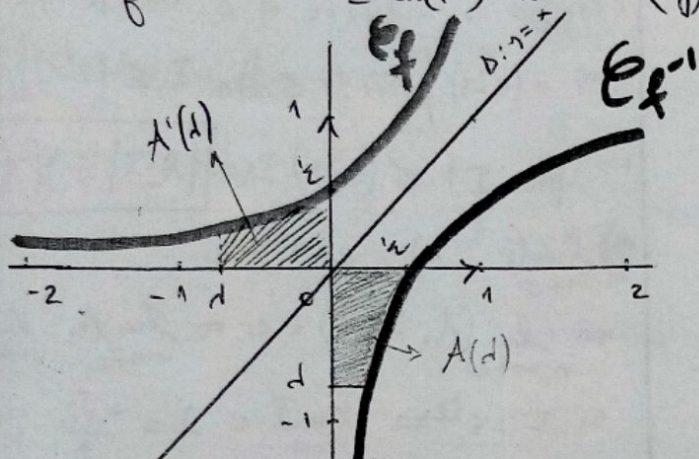


2) $\lim_{x \rightarrow -\infty} f(x) = 0 \Rightarrow y=0 : A.H \text{ en } (-\infty)$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{e^{2x}}{x(1+e^x)} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \rightarrow +\infty} \frac{e^x}{x(e^x+1)}$$

$$= +\infty \text{ car } \lim_{x \rightarrow +\infty} \frac{e^x}{x} = +\infty$$

$\Rightarrow C_f$ admet une BI en $(+\infty)$ de directrice (Oy)



3) a) $f \subseteq \text{strict } \mathbb{R} \text{ sur } \mathbb{R} \Rightarrow f \text{ réalise une bijection de } \mathbb{R} \text{ sur } f(\mathbb{R}) =]0, 1[$

b) $C_f^{-1} = S_D(C_f) / D: y=x$

c) f' est définie par: $\begin{cases} \forall y \in \mathbb{R} \\ f(y) = x \end{cases} \Leftrightarrow \begin{cases} \forall x > 0 \\ y = f^{-1}(x) \end{cases}$

en effet: $f(y) = x \Leftrightarrow \frac{e^{2y}}{1+e^y} = x \Leftrightarrow$

$$e^{2y} = x + x e^y \Leftrightarrow (e^y)^2 - x e^y - x = 0$$

$$\Delta = x^2 + 4x > 0 \text{ car } x > 0$$

$$e^y = \frac{x - \sqrt{x^2 + 4x}}{2} < 0 \text{ à rejeter}$$

$$e^y = \frac{x + \sqrt{x^2 + 4x}}{2} > 0 \Rightarrow y = \ln\left(\frac{x + \sqrt{x^2 + 4x}}{2}\right)$$

concl: $f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2 + 4x}}{2}\right) / x > 0$

4) a) $e^x - \frac{e^x}{1+e^x} = \frac{e^x(1+e^x) - e^x}{1+e^x} = \frac{e^{2x}}{1+e^x} = f(x)$

b) $A(d)$: l'aire limitée par $\begin{cases} C_f^{-1} \\ x=0 \\ y=1 \\ y=0 \end{cases}$

ou posé $A'(d) = S_D(A(d))$

$\Rightarrow A'(d)$: l'aire limitée par $\begin{cases} S_D(C_f^{-1}) \\ S_D(x=0) = (y=0) \\ S_D(y=1) = (x=d) \\ S_D(y=0) = (x=0) \end{cases}$

$$\Rightarrow A'(d) = \int_d^0 f(x) dx = \int_d^0 \left(e^x - \frac{e^x}{1+e^x} \right) dx$$

$$\Rightarrow A'(d) = \left[e^x - \ln(1+e^x) \right]_d^0 = 1 - \ln 2 - e^d + \ln(1+e^d)$$

Comme S_D : isotherme $\Rightarrow$ conserve les aires

$$\Rightarrow A(d) = A'(d) = 1 - \ln 2 - e^d + \ln(1+e^d) \quad (uq)$$

5) $F_n(x) = \int_x^0 \frac{e^{nt}}{1+e^t} dt$ tel que $x \leq 0$ et $n \in \mathbb{N}^*$

1) a) $F_1(x) = \int_x^0 \frac{e^t}{1+e^t} dt = \left[\ln(1+e^t) \right]_x^0 = \ln 2 - \ln(1+e^x)$

$$\lim_{x \rightarrow -\infty} e^x = 0 \Rightarrow \lim_{x \rightarrow -\infty} F_1(x) = \ln 2$$

b) $F_2(x) = \int_x^0 \frac{e^{2t}}{1+e^t} dt = \int_x^0 f(t) dt = A(x)$

$$\Rightarrow F_2(x) = 1 - \ln 2 - e^x + \ln(1+e^x)$$

$$\Rightarrow \lim_{x \rightarrow -\infty} F_2(x) = 1 - \ln 2$$

2) a) $F_{n+1}(x) + F_n(x) = \int_x^0 \frac{e^{(n+1)t}}{1+e^t} dt + \int_x^0 \frac{e^{nt}}{1+e^t} dt$

$$= \int_x^0 \frac{e^{(n+1)t} + e^{nt}}{1+e^t} dt = \int_x^0 \frac{e^{nt}(e^t + 1)}{1+e^t} dt$$

$$\Leftrightarrow F_{n+1}(x) + F_n(x) = \int_x^0 e^{nt} dt = \left[\frac{1}{n} e^{nt} \right]_x^0$$

$$\Rightarrow F_{n+1}(x) + F_n(x) = \frac{1}{n} (1 - e^{nx})$$

b) Dq $\lim_{x \rightarrow -\infty} F_n(x)$: existe then n^*

• pour $n=1$, $\lim_{x \rightarrow -\infty} F_1(x) = \ln 2 \Rightarrow$ vérifiée

Supp p. $\lim_{x \rightarrow -\infty} F_n(x) = l$ (fini)

• Dq $\lim_{x \rightarrow -\infty} F_{n+1}(x) = l'$ (fini)?

en effet: $F_{n+1}(x) = -F_n(x) + \frac{1}{n} (1 - e^{nx})$

a) $\lim_{x \rightarrow -\infty} F_n(x) = l \Rightarrow \lim_{x \rightarrow -\infty} F_{n+1}(x) = -l + \frac{1}{n}$

d) $\lim_{x \rightarrow -\infty} \frac{1}{n} (1 - e^{nx}) = \frac{1}{n} \Rightarrow$ existe et fini

concl: then n^* , $\lim_{x \rightarrow -\infty} F_n(x)$: est fini.

$$3) a) 2e^t - (1+e^t) = e^t - 1 \leq 0 \text{ car } t \leq 0$$

$$\Rightarrow 2e^t \leq 1+e^t \quad (1)$$

$$b) (1+e^t) - 2 = e^t - 1 \leq 0 \text{ car } t \leq 0$$

$$\Rightarrow 1+e^t \leq 2 \quad (2)$$

$$(1) \wedge (2) \Rightarrow \forall t \leq 0, \boxed{2e^t \leq 1+e^t \leq 2}$$

$$b) \frac{1}{2} \leq \frac{1}{1+e^t} \leq \frac{1}{2e^t}$$

$$\text{car } e^{nt} > 0 \Rightarrow \frac{1}{2} e^{nt} \leq \frac{e^{nt}}{1+e^t} \leq \frac{e^{(n-1)t}}{2}$$

$$\text{Généraliser } x \leq 0 \Rightarrow \frac{1}{2} \int_x^0 e^{nt} dt \leq \int_x^0 \frac{e^{nt}}{1+e^t} dt \leq \frac{1}{2} \int_x^0 e^{(n-1)t} dt$$

$$\Rightarrow \frac{1}{2} \left[\frac{1}{n} e^{nt} \right]_x^0 \leq F_n(x) \leq \frac{1}{2} \left[\frac{1}{n-1} e^{(n-1)t} \right]_x^0$$

$$\Rightarrow \boxed{\frac{1}{2n} (1 - e^{nx}) \leq F_n(x) \leq \frac{1}{2(n-1)} (1 - e^{(n-1)x})}$$

$$c) \lim_{x \rightarrow -\infty} \frac{1}{2n} (1 - e^{nx}) = \frac{1}{2n}$$

$$\lim_{x \rightarrow -\infty} F_n(x) = R_n$$

$$\lim_{x \rightarrow -\infty} \frac{1}{2(n-1)} (1 - e^{(n-1)x}) = \frac{1}{2(n-1)}$$

$$\Rightarrow \boxed{\frac{1}{2n} \leq R_n \leq \frac{1}{2(n-1)}}$$

$$d) G_n(x) = (-1)^n \int_x^0 e^{nt} dt \quad \forall x \leq 0 \text{ et } n \in \mathbb{N}^*$$

$$e) G_n(x) = (-1)^n \left[\frac{1}{n} e^{nt} \right]_x^0 = \frac{(-1)^n}{n} (1 - e^{nx})$$

$$\lim_{x \rightarrow -\infty} e^{nx} = 0 \text{ et } \frac{(-1)^n}{n} \cdot n \Rightarrow \boxed{\lim_{x \rightarrow -\infty} G_n(x) = \frac{(-1)^n}{n}}$$

$$b) \sum_{k=1}^n G_k(x) = \sum_{k=1}^n (-1)^k \int_x^0 e^{kt} dt$$

$$= \int_x^0 \sum_{k=1}^n (-1)^k e^{kt} dt = \int_x^0 \sum_{k=1}^n (-e^t)^k dt$$

$$= \int_x^0 (-e^t) \left[\frac{1 - (-e^t)^{n+1}}{1 - (-e^t)} \right] dt = \int_x^0 \frac{-e^t - (-e^t)^{n+1}}{1 + e^t} dt$$

$$= \int_x^0 \frac{-e^t}{1 + e^t} dt - (-1)^{n+1} \int_x^0 \frac{e^{(n+1)t}}{1 + e^t} dt$$

$$= -F_1(x) + (-1)^n F_{n+1}(x)$$

$$\Rightarrow G_1(x) + G_2(x) + \dots + G_n(x) = -F_1(x) + (-1)^n F_{n+1}(x)$$

$$d) \text{th} \in \mathbb{N}^*, U_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} = - \sum_{k=1}^n \frac{(-1)^k}{k}$$

$$\text{or } \lim_{x \rightarrow -\infty} G_k(x) = \frac{(-1)^k}{k}$$

$$\Rightarrow \sum_{k=1}^n \lim_{x \rightarrow -\infty} G_k(x) = \lim_{x \rightarrow -\infty} \sum_{k=1}^n G_k(x) = \sum_{k=1}^n \frac{(-1)^k}{k}$$

$$\Rightarrow U_n = - \lim_{x \rightarrow -\infty} \sum_{k=1}^n G_k(x)$$

$$\text{or } \sum_{k=1}^n G_k(x) = -F_1(x) + (-1)^n F_{n+1}(x)$$

$$\Rightarrow U_n = - \lim_{x \rightarrow -\infty} [-F_1(x) + (-1)^n F_{n+1}(x)]$$

$$U_n = \ln 2 - (-1)^n R_{n+1}$$

$$\Rightarrow \boxed{U_n = \ln 2 + (-1)^{n+1} R_{n+1}}$$

$$b) \text{d'après B) 3) c) } \frac{1}{2n} \leq R_n \leq \frac{1}{2(n-1)}$$

$$\Rightarrow \frac{1}{2(n+1)} \leq R_{n+1} \leq \frac{1}{2n}$$

$$\lim_{n \rightarrow +\infty} \frac{1}{2(n+1)} = \lim_{n \rightarrow +\infty} \frac{1}{2n} = 0 \Rightarrow \lim_{n \rightarrow +\infty} R_{n+1} = 0$$

$$\text{or } (-1)^{n+1} = 1 \Rightarrow |(-1)^{n+1} R_{n+1}| = |R_{n+1}|$$

$$\text{et comme } \lim_{n \rightarrow +\infty} R_{n+1} = 0 \Rightarrow \lim_{n \rightarrow +\infty} (-1)^{n+1} R_{n+1} = 0$$

$$\text{alors } \boxed{\lim_{n \rightarrow +\infty} U_n = \ln 2} \text{ ainsi } U_n \text{ converge vers } \ln 2$$

Problème 06: BAC 2004 - Session Principale

$$A) f_1(x) = \sqrt{1+x} \cdot e^{-x} \quad \mathcal{D}_{f_1} = [-1, +\infty[$$

$$1) a) \text{ dérivabilité de } f_1 \text{ en } (-1)^+ : f_1(-1) = 0$$

$$\lim_{x \rightarrow (-1)^+} \frac{f_1(x) - f_1(-1)}{x - (-1)} = \lim_{x \rightarrow (-1)^+} \frac{\sqrt{1+x} \cdot e^{-x}}{1+x}$$

$$= \lim_{x \rightarrow (-1)^+} \frac{(1+x) e^{-x}}{(1+x) \sqrt{1+x}} = \lim_{x \rightarrow (-1)^+} \frac{e^{-x}}{\sqrt{1+x}} = \frac{e}{0^+} = +\infty \Rightarrow f_1 \text{ n'est pas}$$

dérivable en $(-1)^+$ et f_1 admet en $(-1, 0)$ une demi-tangente verticale dirigée vers le haut

$$b) \lim_{x \rightarrow +\infty} f_1(x) = \lim_{x \rightarrow +\infty} \sqrt{1+x} \cdot e^{-x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1+x}}{e^x}$$

$$= \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{1+x}}{x}}{\frac{e^x}{x}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{\frac{1}{x} + \frac{1}{x^2}}}{\frac{e}{x}} = \frac{0}{+\infty} = 0$$

9) $y=0$ est une asymptote horizontale à la courbe $\mathcal{C}_1$ au voisinage de $(+\infty)$

2) $f_1 \in \mathcal{C}^1$ sur $1+x > 0 \Rightarrow f_1 \in \mathcal{C}^1$ sur $]-1, +\infty[$

$$f_1'(x) = \frac{1}{2\sqrt{1+x}} e^{-x} - \sqrt{1+x} e^{-x} = \frac{e^{-x} - 2(1+x)e^{-x}}{2\sqrt{1+x}}$$

$$\Rightarrow f_1'(x) = \frac{(-1-2x)e^{-x}}{2\sqrt{1+x}}, f_1'(x)=0 \Leftrightarrow x = -\frac{1}{2}$$

$$\text{sgn}(f_1'(x)) = \text{sgn}(-1-2x)$$

x	-1	$-\frac{1}{2}$	0	$+\infty$
$f_1'(x)$		+	0	-
$f_1(x)$			$\sqrt{\frac{e}{2}}$	0

$f(-\frac{1}{2}) = \sqrt{\frac{e}{2}} = \sqrt{\frac{e}{2}} \approx 1,2$

3) $T_0: y = f_1'(0)(x-0) + f_1(0) \Rightarrow$
 $\alpha f_1'(0) = -\frac{1}{2}$ et $f_1(0) = 1$

$$T_0: y = -\frac{1}{2}x + 1$$

4) on pose $\varphi(x) = f_1(x) - x$, $\mathcal{D}\varphi = [-\frac{1}{2}, +\infty[$
 $\varphi \in \mathcal{C}^1$ sur $[-\frac{1}{2}, +\infty[$, $\varphi'(x) = f_1'(x) - 1 < 0$
 car $f_1'(x) \leq 0$ sur $[-\frac{1}{2}, +\infty[$

$$\lim_{x \rightarrow +\infty} \varphi(x) = \lim_{x \rightarrow +\infty} (f_1(x) - x) = -\infty$$

x	$-\frac{1}{2}$	α	$+\infty$
$\varphi'(x)$		-	
$\varphi(x)$	$\sqrt{\frac{e}{2}} + \frac{1}{2}$		$-\infty$

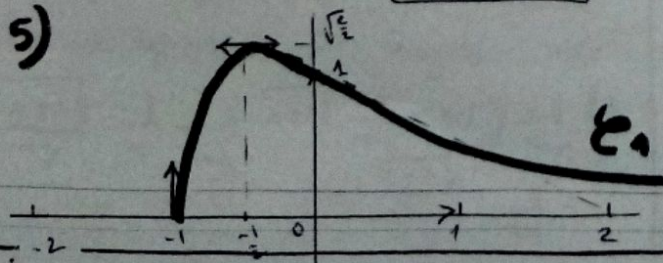
$\varphi(-\frac{1}{2}) = f_1(-\frac{1}{2}) - (-\frac{1}{2}) = \sqrt{\frac{e}{2}} + \frac{1}{2}$
 $\varphi \in \mathcal{C}^1$ sur $[-\frac{1}{2}, +\infty[$
 et $\varphi(-\frac{1}{2}, +\infty) =]-\infty, \sqrt{\frac{e}{2}} + \frac{1}{2}]$
 $\alpha \in]-\infty, \sqrt{\frac{e}{2}} + \frac{1}{2}]$

$$\Rightarrow \exists! \alpha \in]-\frac{1}{2}, +\infty[/ \varphi(x) = 0 \Leftrightarrow f_1(x) = \alpha$$

$$\varphi(-\frac{1}{2}) = f_1(-\frac{1}{2}) - (-\frac{1}{2}) = \sqrt{\frac{e}{2}} + \frac{1}{2} = \sqrt{\frac{3}{2}} - \frac{1}{2} \approx 0,24$$

$$\varphi(1) = f_1(1) - 1 = \sqrt{2}e^{-1} - 1 \approx -0,48 < 0$$

$$\Rightarrow \varphi(-\frac{1}{2}) \cdot \varphi(1) < 0 \Rightarrow \boxed{\frac{1}{2} < \alpha < 1}$$



6) a) on pose $g(x) = 1+x-e^x$

$$\mathcal{D}g = \mathbb{R}$$

$$g \in \mathcal{C}^1$$
 sur $\mathbb{R}$, $g'(x) = 1 - e^x$, $g'(x) = 0 \Leftrightarrow x = 0$

x	$-\infty$	0	$+\infty$
$g'(x)$	+	0	-
$g(x)$		0	

$g(0) = 0$: maximum absolu
 $\Rightarrow \forall x \in \mathbb{R}, g(x) \leq 0$
 $\Rightarrow 1+x-e^x \leq 0$
 $\Rightarrow \boxed{1+x \leq e^x}$

b) pm $x \geq -1 \Rightarrow 0 \leq 1+x \leq e^x$

$$\Rightarrow \sqrt{1+x} \leq \sqrt{e^x} = e^{\frac{x}{2}}$$

$$e^x > 0 \Rightarrow \sqrt{1+x} e^{-x} \leq e^{\frac{x}{2}} e^{-x} \Rightarrow \boxed{f_1(x) \leq e^{-\frac{x}{2}}}$$

7) $d \geq 1$, $S(d) = \int_1^d f_1(x) dx$

8) $S(d)$: l'aire de la partie du plan limitée par la courbe $\mathcal{C}_1$, l'axe des abscisses et les droites $x=1$ et $x=d$

9) on a $0 \leq f_1(x) \leq e^{-\frac{x}{2}}$
 pm $1 \leq d \Rightarrow 0 \leq \int_1^d f_1(x) dx \leq \int_1^d e^{-\frac{x}{2}} dx$
 $\Rightarrow 0 \leq S(d) \leq [-2e^{-\frac{x}{2}}]_1^d = -2e^{-\frac{d}{2}} + 2e^{-\frac{1}{2}}$
 $\Rightarrow -2e^{-\frac{d}{2}} < 0 \Rightarrow \boxed{0 \leq S(d) \leq \frac{2}{\sqrt{e}}}$

10) $f_n(x) = \sqrt{1+x} \cdot e^{-\frac{x}{n}}$ $\forall n \in \mathbb{N}^*$

1) Points fixes: $f_n(x) = f_{n'}(x) / n \neq n'$

$$\Rightarrow \sqrt{1+x} e^{-\frac{x}{n}} = \sqrt{1+x} e^{-\frac{x}{n'}} \Rightarrow \sqrt{1+x} (e^{-\frac{x}{n}} - e^{-\frac{x}{n'}}) = 0$$

$$\Leftrightarrow \begin{cases} \sqrt{1+x} = 0 \Rightarrow x = -1 \Rightarrow y = 0 \Rightarrow A(-1, 0) \\ \text{ou} \\ e^{-\frac{x}{n}} = e^{-\frac{x}{n'}} \Rightarrow -\frac{x}{n} = -\frac{x}{n'} \Rightarrow \end{cases}$$

$$-xn = -xn' \Rightarrow x(n-n') = 0 \Rightarrow \begin{cases} x=0 \\ n=n' \end{cases}$$

$$\Rightarrow y = 1 \Rightarrow B(0, 1)$$

$$\text{Concl: } \{A(-1, 0), B(0, 1)\} \subset \mathcal{C}_n \forall n \in \mathbb{N}^*$$

2) a) $\forall x > -1$, $f_n'(x) = \frac{1}{2\sqrt{1+x}} e^{-\frac{x}{n}} - \frac{\sqrt{1+x}}{n} e^{-\frac{x}{n}}$

$$\Leftrightarrow f_n'(x) = \frac{e^{-\frac{x}{n}}}{2n\sqrt{1+x}} (n - 2(1+x))$$

$$\Rightarrow f_n'(x) = \frac{e^{-\frac{x}{n}}}{2n\sqrt{1+x}} (-2x + (n-2))$$

$$f_n'(x) = 0 \Leftrightarrow -2x + n - 2 = 0 \Leftrightarrow x = -1 + \frac{n}{2} > -1$$

$\Rightarrow E_n$ admet en M_n d'abscisse $(-1 + \frac{1}{2^n})$ une tangente horizontale $\forall n \in \mathbb{N}^*$

b) $\alpha_n = -1 + \frac{1}{2^n} \Rightarrow \alpha_n$ est le terme général de la suite arithmétique définie

par $\alpha_1 = -\frac{1}{2}$ et $r = \frac{1}{2}$

3) Posons de E_n et E_{n+1} :

on pose $h(x) = f_n(x) - f_{n+1}(x) =$

$$\Rightarrow h(x) = \sqrt{1+x} \left(e^{-\frac{x}{n}} - e^{-\frac{x}{n+1}} \right)$$

$$\Rightarrow h(x) = \sqrt{1+x} e^{-\frac{x}{n}} \left(1 - e^{-\frac{x}{n+1} + \frac{x}{n}} \right)$$

$$\Rightarrow h(x) = \sqrt{1+x} e^{-\frac{x}{n}} \left(1 - e^{\frac{x}{n(n+1)}} \right)$$

on veut $h(x) = 0 \Leftrightarrow$ $\begin{cases} x = -1 \\ \text{ou} \\ e^{\frac{x}{n(n+1)}} = 1 \Leftrightarrow x = 0 \end{cases}$

x	-1	0	
f(x)	0	+	-
Position de E_n par rapport à E_{n+1}	$E_n = E_{n+1}$	E_n au dessus de E_{n+1}	E_n au dessous de E_{n+1}

4) $I = \int_{-1}^1 \sqrt{1+x} dx$, $A_n = \int_{-1}^1 f_n(x) dx$

a) $I = \int_{-1}^1 (1+x)^{\frac{1}{2}} dx = \left[\frac{2}{3} (1+x)^{\frac{3}{2}} \right]_{-1}^1 = \frac{2}{3} 2\sqrt{2} - 0$

$\Rightarrow I_1 = \frac{4\sqrt{2}}{3}$

b) $(1 - e^{-\frac{1}{n}}) - (e^{-\frac{1}{n+1}} - 1) = -e^{-\frac{1}{n}} + 2 - e^{-\frac{1}{n+1}}$
 $= -e^{-\frac{1}{n}} \left[(e^{-\frac{1}{n+1}})^2 - 2e^{-\frac{1}{n+1}} + 1 \right] = -e^{-\frac{1}{n}} (e^{-\frac{1}{n+1}} - 1)^2 \leq 0$
 $\Rightarrow 1 - e^{-\frac{1}{n}} \leq e^{-\frac{1}{n+1}} - 1$

c) $-1 \leq x \leq 1 \Leftrightarrow -1 \leq -x \leq 1$

$\Rightarrow -\frac{1}{n} \leq -\frac{x}{n} \leq \frac{1}{n}$

$e^{-\frac{1}{n}} \leq e^{-\frac{x}{n}} \leq e^{\frac{1}{n}}$

$\Rightarrow -e^{-\frac{1}{n}} \leq -e^{-\frac{x}{n}} \leq -e^{\frac{1}{n}}$

$\Rightarrow 1 - e^{-\frac{1}{n}} \leq 1 - e^{-\frac{x}{n}} \leq 1 - e^{\frac{1}{n}}$

d'où $e^{-\frac{1}{n}} - 1 \leq e^{-\frac{x}{n}} - 1 \leq e^{\frac{1}{n}} - 1$

ou $1 - e^{-\frac{1}{n}} \leq e^{-\frac{x}{n}} - 1$

$1 - e^{-\frac{1}{n}} \leq e^{-\frac{x}{n}} - 1 \leq e^{\frac{1}{n}} - 1$

$\Rightarrow |e^{-\frac{x}{n}} - 1| \leq e^{\frac{1}{n}} - 1$ d'où $|1 - e^{-\frac{x}{n}}| \leq e^{\frac{1}{n}} - 1$

d) $A_n - I = \int_{-1}^1 f_n(x) dx - \int_{-1}^1 \sqrt{1+x} dx$
 $= \int_{-1}^1 (f_n(x) - \sqrt{1+x}) dx$

$\Rightarrow A_n - I = \int_{-1}^1 \sqrt{1+x} (e^{-\frac{x}{n}} - 1) dx$

on $|e^{-\frac{x}{n}} - 1| \leq e^{\frac{1}{n}} - 1$

$-(e^{\frac{1}{n}} - 1) \leq e^{-\frac{x}{n}} - 1 \leq e^{\frac{1}{n}} - 1$

$\sqrt{1+x} \geq 0 \Rightarrow -\sqrt{1+x} (e^{\frac{1}{n}} - 1) \leq \sqrt{1+x} (e^{-\frac{x}{n}} - 1) \leq \sqrt{1+x} (e^{\frac{1}{n}} - 1)$

$\Rightarrow -(e^{\frac{1}{n}} - 1) \int_{-1}^1 \sqrt{1+x} dx \leq A_n - I \leq (e^{\frac{1}{n}} - 1) \int_{-1}^1 \sqrt{1+x} dx$

$\Rightarrow |A_n - I| \leq (e^{\frac{1}{n}} - 1) I \Rightarrow |A_n - I| \leq \frac{4\sqrt{2}}{3} (e^{\frac{1}{n}} - 1)$

e) $\lim_{n \rightarrow +\infty} (e^{\frac{1}{n}} - 1) = e^0 - 1 = 0$

$\Rightarrow \lim_{n \rightarrow +\infty} (A_n - I) = 0 \Rightarrow \lim_{n \rightarrow +\infty} A_n = \lim_{n \rightarrow +\infty} I$

on $I = \frac{4\sqrt{2}}{3} \Rightarrow \lim_{n \rightarrow +\infty} I = I = \frac{4\sqrt{2}}{3}$

$\Rightarrow \lim_{n \rightarrow +\infty} A_n = \frac{4\sqrt{2}}{3}$

concl: A_n converge vers $\frac{4\sqrt{2}}{3}$

Exercice 7: BAC 2004 - Session de Contrôle

$f(x) = \ln(x^2 - 2x + 2)$ Df = $\mathbb{R}$

1) a) $f \in \mathcal{C}^1$ sur $\mathbb{R}$ car $x^2 - 2x + 2 = (x-1)^2 + 1 > 0$

$f'(x) = \frac{2x-2}{x^2-2x+2} \Rightarrow f'(x) = 0 \Leftrightarrow x = 1$

x	$-\infty$	1	$+\infty$
f(x)	$+\infty$	0	$+\infty$
f'(x)	$+\infty$	0	$+\infty$

$f(1) = \ln 1 = 0$
 $\lim_{x \rightarrow \pm\infty} f(x) = +\infty$
 car $\lim_{x \rightarrow \pm\infty} (x^2 - 2x + 2) = +\infty$

b) 1) a) D: $x=1$ Axe de symétrie par $\in$

car $(2-x) \in \mathbb{R}$ et $f(2-x) = \ln((2-x)^2 - 2(2-x) + 2)$

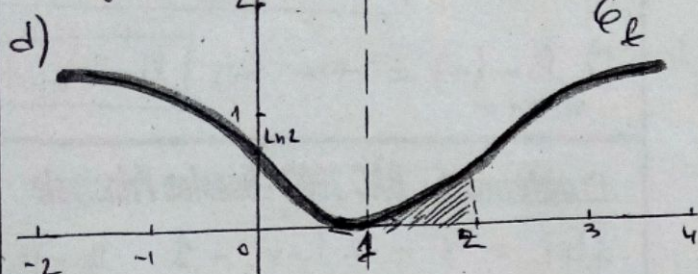
$\Rightarrow f(2-x) = \ln(4 - 4x + x^2 - 4 + 2x + 2) = \ln(x^2 - 2x + 2)$

$\Rightarrow f(2-x) = f(x)$ d'où D: $x=1$: axe de sym.

$$c) \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{t \rightarrow \pm\infty} \frac{\ln(x^2 - 2x + 2)}{x} = \lim_{t \rightarrow \pm\infty} \frac{\ln(x^2(1 - \frac{2}{x} + \frac{2}{x^2}))}{x}$$

$$= \lim_{t \rightarrow \pm\infty} \frac{\ln(x^2)}{x} + \lim_{t \rightarrow \pm\infty} \frac{\ln(1 - \frac{2}{x} + \frac{2}{x^2})}{x} = 0$$

$\Rightarrow$ f admet 2 BI en $(\pm\infty)$ de dir $(0, \infty)$



$$2) f(x) = \int_1^{1+\sqrt{x}} \frac{dt}{t^2 - 2t + 2} \quad D_f = [0, \frac{\pi}{2}]$$

a) on pose $g: t \mapsto \frac{1}{t^2 - 2t + 2} \in \text{fon } \mathbb{R}$

$\Rightarrow g$ admet une primitive $G \in \text{fon } \mathbb{R} / G'(t) = g(t)$

$$\Rightarrow f(x) = [G(t)]_1^{1+\sqrt{x}} = G(1+\sqrt{x}) - G(1)$$

or $x \mapsto \sqrt{x} \in \text{fon } [0, \frac{\pi}{2}]$
et $\forall x \in [0, \frac{\pi}{2}], (1+\sqrt{x}) \in \mathbb{R} = D_g \Rightarrow f \in \text{fon } [0, \frac{\pi}{2}]$

$$f'(x) = [G(1+\sqrt{x})]' - [G(1)]' \\ = (1+\sqrt{x}) \cdot g(1+\sqrt{x}) - 0 = \frac{(1+\sqrt{x})}{(1+\sqrt{x})^2 - 2(1+\sqrt{x}) + 2} \cdot 1 \\ = \frac{1+\sqrt{x}}{1+2\sqrt{x}+\sqrt{x}^2 - 2-2\sqrt{x}+2} = \frac{1+\sqrt{x}}{1+\sqrt{x}^2} = 1$$

$$\Rightarrow \boxed{f'(x) = 1}$$

b) $f'(x) = 1 \Rightarrow f(x) = x + h$

or $f(0) = \int_1^1 g(t) dt = 0 \Rightarrow h = 0$ d'où $\boxed{f(x) = x}$

$$f\left(\frac{\pi}{4}\right) = \int_1^{1+\sqrt{\frac{\pi}{4}}} g(t) dt = \int_1^2 \frac{dt}{t^2 - 2t + 2} = \frac{\pi}{4}$$

alors $\boxed{\int_1^2 \frac{dt}{t^2 - 2t + 2} = \frac{\pi}{4}}$

3) a) st $I = \int_1^2 f(x) dx = \int_1^2 \ln(x^2 - 2x + 2) dx$

on pose $u(x) = \ln(x^2 - 2x + 2) \rightarrow u'(x) = \frac{2x-2}{x^2-2x+2}$
 $v'(x) = 1 \rightarrow v(x) = x$

$$\Rightarrow I = \left[x \cdot \ln(x^2 - 2x + 2) \right]_1^2 - 2 \int_1^2 \frac{x^2 - x}{x^2 - 2x + 2} dx$$

$$\Rightarrow I = 2 \ln 2 - 2 \int_1^2 \frac{(x^2 - x)}{(x^2 - 2x + 2)} dx$$

b) $1 + \frac{x-1}{x^2-2x+2} - \frac{1}{x^2-2x+2} = \frac{(x^2-2x+2) + (x-1) - x}{x^2-2x+2}$

$$\Rightarrow \frac{x^2 - x}{x^2 - 2x + 2} = 1 + \frac{x-1}{x^2-2x+2} - \frac{1}{x^2-2x+2}$$

$$c) A = \int_1^2 f(x) dx = 2 \ln 2 - 2 \int_1^2 \left(\frac{x^2 - x}{x^2 - 2x + 2} \right) dx$$

$$\Rightarrow A = 2 \ln 2 - 2 \int_1^2 \left(\frac{-1}{x^2 - 2x + 2} + \frac{x-1}{x^2 - 2x + 2} + 1 \right) dx$$

$$\Rightarrow A = 2 \ln 2 + 2 \int_1^2 \frac{dk}{k^2 - 2k + 2} - \int_1^2 \frac{2x-2}{x^2-2x+2} dx - 2 \int_1^2 1 dx \\ = 2 \ln 2 + 2 \left(\frac{\pi}{4} \right) - \left[\ln(x^2 - 2x + 2) \right]_1^2 - 2[x]_1^2$$

$$\Rightarrow A = 2 \ln 2 + \frac{\pi}{2} - \ln 2 + \ln 1 - 2(2-1)$$

$$\Rightarrow \boxed{A = \ln 2 + \frac{\pi}{2} - 2} \text{ (ua)}$$

Problème 8 : BAC 2003 - Session Principale

I) 1) $h(x) = x - \ln x \quad D_h =]0, +\infty[$

a) $h \in \text{fon }]0, +\infty[\quad h'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$

x	0	1	$+\infty$
h'	-	0	+
h	$+\infty$	1	$+\infty$

$\lim_{x \rightarrow 0^+} h(x) = \lim_{x \rightarrow 0^+} x - \ln x = +\infty$
 $\lim_{x \rightarrow +\infty} h(x) = \lim_{x \rightarrow +\infty} x - \ln x = +\infty$
 $h(1) = 1$

b) $h(1) = 1$: minimum absolu de f
 $\Rightarrow \forall x > 0, \quad \boxed{h(x) \geq 1}$

2) $f(x) = \frac{1}{x - \ln x}$ si $x > 0$ et $f(0) = 0$

a) or $\forall x > 0, h(x) \geq 1 \Rightarrow h(x) \neq 0$

$\Rightarrow f \in \text{fon }]0, +\infty[$

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x - \ln x} = \lim_{x \rightarrow 0^+} \frac{1}{h(x)} = \frac{1}{+\infty} = 0$

$\Rightarrow f \text{ est en } 0^+ \text{ Cml. } D_f = [0, +\infty[$

b) $\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{1}{x(x - \ln x)}$

$$= \lim_{x \rightarrow 0^+} \frac{1}{x^2 \ln x} = \frac{1}{0^+} = +\infty$$

$\Rightarrow f$ n'est pas $\in$ en 0^+ et f admet en $0(0,0)$ une demi-tangente verticale dirigée haut

II) $F(x) = \int_x^{2x} f(t) dt \quad D_f = [0, +\infty[$

1) a) f étant $\in \mathcal{R}_m[0, +\infty[\Rightarrow f$ admet une primitive $\varphi \in \mathcal{R}_m[0, +\infty[/ \varphi'(t) = f(t)$

$$\text{donc } F(x) = [\varphi(t)]_x^{2x} = \varphi(2x) - \varphi(x)$$

$$\text{or } \varphi \in \mathcal{R}_m \mathbb{R}_+$$

$$\text{et } x \mapsto 2x \in \mathcal{R}_m \mathbb{R}_+ \text{ et } (2x) \in \mathbb{R}_+$$

$$\Rightarrow \boxed{F \in \mathcal{R}_m \mathbb{R}_+}$$

$$b) \forall x \in \mathbb{R}_+; F'(x) = [\varphi(2x)]' - [\varphi(x)]'$$

$$\Rightarrow F'(x) = 2\varphi'(2x) - 1 \cdot \varphi'(x)$$

$$\Rightarrow F'(x) = 2f(2x) - f(x)$$

$$\cdot \text{Si } x > 0 \Rightarrow F'(x) = \frac{2}{2x - \ln(2x)} - \frac{1}{x - \ln x}$$

$$\Rightarrow F'(x) = \frac{2x - 2\ln x - 2x + \ln 2x}{h(2x) \cdot h(x)}$$

$$\Rightarrow F'(x) = \frac{-2\ln x + \ln 2 + \ln x}{h(2x) \cdot h(x)}$$

$$\Rightarrow \forall x > 0, F'(x) = \frac{\ln 2 - \ln x}{h(2x) \cdot h(x)}$$

$$\cdot \text{Si } x = 0, F'(0) = 2f(0) - f(0) = 0$$

$$\Rightarrow \int_x^{2x} \frac{1}{t} dt = [\ln t]_x^{2x} = \ln(2x) - \ln x$$

$$= \ln\left(\frac{2x}{x}\right) = \ln 2 \Rightarrow \int_x^{2x} \frac{dt}{t} = \ln 2$$

$$\Rightarrow \forall x \geq 1, 0 \leq F(x) - \ln 2 \leq \frac{\ln 2x}{x - \ln x}$$

$$\text{or } \frac{1}{t - \ln t} - \frac{1}{t} = \frac{\ln t}{t \cdot h(t)}$$

$$\text{a } x \leq t \leq 2x \text{ et } n \geq 1 \text{ or } h \in \mathcal{R}_m[1, +\infty[$$

$$\Rightarrow h(x) \leq h(t) \leq h(2x)$$

$$x \leq t \leq 2x$$

$$\text{Produit: } x h(x) \leq t h(t) \leq 2x h(2x)$$

$$\text{donc } 0 \leq \frac{1}{t h(t)} \leq \frac{1}{x h(x)}$$

$$\text{et } \ln x \leq \ln t \leq \ln 2x$$

$$\text{Produit: } 0 \leq \frac{\ln t}{t h(t)} \leq \frac{\ln(2x)}{x h(x)}$$

$$\Rightarrow 0 \leq f(t) - \frac{1}{t} \leq \frac{\ln(2x)}{x h(x)}$$

$$\text{donc } 0 \leq \int_x^{2x} f(t) dt - \int_x^{2x} \frac{1}{t} dt \leq \int_x^{2x} \frac{\ln(2x)}{x h(x)} dt$$

$$\Rightarrow 0 \leq F(x) - \ln 2 \leq \frac{\ln(2x)}{x h(x)} \cdot [t]_x^{2x}$$

$$\Rightarrow 0 \leq F(x) - \ln 2 \leq \frac{\ln(2x)}{x - \ln x}$$

$$\cdot \lim_{x \rightarrow +\infty} \frac{\ln(2x)}{x - \ln x} = \lim_{x \rightarrow +\infty} \frac{\frac{\ln(2x)}{2x}}{\frac{x}{2x} - \frac{\ln x}{2x}} = \frac{0}{\frac{1}{2} - 0} = 0$$

$$\Rightarrow \lim_{x \rightarrow +\infty} (F(x) - \ln 2) = 0 \Rightarrow \lim_{x \rightarrow +\infty} F(x) = \ln 2$$

$$b) \text{ or } F\left(\frac{1}{2}\right) = \int_{\frac{1}{2}}^1 \frac{1}{h(t)} dt$$

$$\text{or } h \in \mathcal{R}_m\left[\frac{1}{2}, 1\right] \Rightarrow h(1) \leq h(t) \leq h\left(\frac{1}{2}\right)$$

$$\text{donc } \frac{1}{h(t)} \leq h(1) = 1$$

$$\text{donc } \int_{\frac{1}{2}}^1 \frac{1}{h(t)} dt \leq \int_{\frac{1}{2}}^1 1 dt$$

$$\Rightarrow F\left(\frac{1}{2}\right) \leq [t]_{\frac{1}{2}}^1 = \frac{1}{2} \leq \ln 2$$

$$\Rightarrow \boxed{F\left(\frac{1}{2}\right) \leq \ln 2}$$

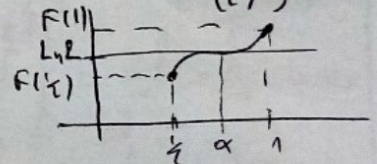
$$b) F \text{ étant } \in \mathcal{R}_m \mathbb{R}_+ \Rightarrow F \in \mathcal{R}_m\left[\frac{1}{2}, 1\right]$$

$$F\left(\frac{1}{2}\right) \leq \ln 2 \text{ et d'après II) 3) or}$$

$$F(1) - \ln(2) \geq 0 \Rightarrow F(1) \geq \ln 2$$

alors il existe au moins $\alpha \in \left(\frac{1}{2}, 1\right)$ tel que

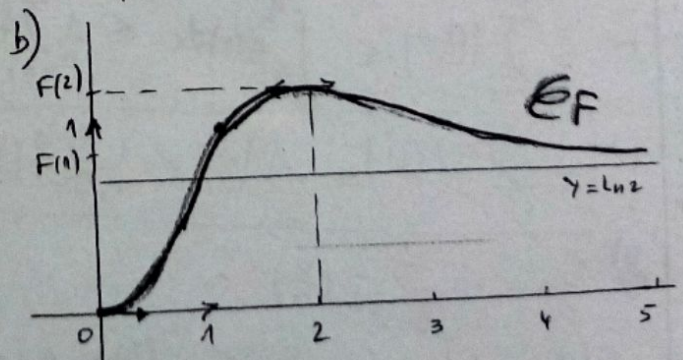
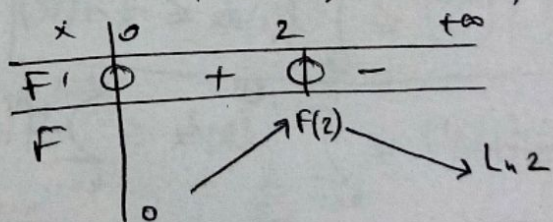
$$\boxed{F(\alpha) = \ln 2}$$



$$c) \text{ a) on résoud: } F'(x) = 0 \text{ et } x > 0$$

$$\Rightarrow \ln 2 - \ln x = 0 \Leftrightarrow \ln x = \ln 2 \Leftrightarrow x = 2$$

$$\text{or } \forall t > 0, h(t) \geq 1 \Rightarrow h(2x) > 0 \text{ et } h(t) > 0$$



$$d) \forall n \in \mathbb{N}^+ \quad V_n = \int_{\frac{1}{n}}^1 \frac{t}{t - \ln t} dt$$

a) Mq $\forall t > 0 \quad \frac{t}{t-\ln t} \leq t$
 or $\forall t > 0, h(t) \geq 1 \Rightarrow \frac{1}{h(t)} \leq 1$
 or $t > 0 \Rightarrow \frac{t}{h(t)} < t \Rightarrow \boxed{\frac{t}{t-\ln t} \leq t}$

b) $V_{n+1} - V_n = \int_{\frac{1}{n+1}}^1 \frac{t}{t-\ln t} dt - \int_{\frac{1}{n}}^1 \frac{t}{t-\ln t} dt$
 $= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{t}{t-\ln t} dt > 0$

car $\frac{t}{h(t)} > 0$ et $\frac{1}{n+1} < \frac{1}{n} \Rightarrow V_{n+1} - V_n > 0$

d'où V_n strict $\nearrow$ sur $\mathbb{N}^*$

c) on a $\frac{t}{t-\ln t} \leq t \Rightarrow 0 \leq \int_{\frac{1}{n}}^1 \frac{t}{t-\ln t} dt \leq \int_{\frac{1}{n}}^1 t dt$
 $\Leftrightarrow 0 \leq V_n \leq \left[\frac{1}{2} t^2 \right]_{\frac{1}{n}}^1 = \frac{1}{2} - \frac{1}{2n^2} \leq \frac{1}{2}$
 $\Rightarrow V_n$ est majorée par $\frac{1}{2} \Rightarrow V_n$ converge
 or $V_n \nearrow$

et converge vers $l \in [0, \frac{1}{2}]$

2) $W_n = \int_1^n \frac{t}{t-\ln t} dt \quad \forall n \in \mathbb{N}^*$

a) $\frac{t}{t-\ln t} - (1+\ln t) = \frac{t - (1+\ln t)(t-\ln t)}{t-\ln t}$
 $= \frac{t - t + t\ln t - t\ln t + (\ln t)^2}{t-\ln t}$
 $= \frac{(\ln t)^2}{t-\ln t}$
 $= \frac{\ln t \cdot [1 - (t-\ln t)]}{t-\ln t} = \frac{\ln t \cdot [1-h(t)]}{t-\ln t}$

or $t \geq 1 \Rightarrow \ln(t) > 0$ et $h(t) \geq 1 \Rightarrow 1-h(t) \leq 0$
 $\Rightarrow \frac{\ln t \cdot [1-h(t)]}{t-\ln t} \leq 0 \Rightarrow \boxed{\frac{t}{t-\ln t} \leq 1+\ln t}$

alors $\int_1^n \frac{t}{t-\ln t} dt \leq \int_1^n (1+\ln t) dt$

$\Leftrightarrow W_n \leq \left[t + t\ln t - t \right]_1^n = n \ln(n)$

car $\boxed{W_n \leq n \ln(n)}$

b) $\int_1^n \left(1 + \frac{\ln t}{t}\right) dt = \left[t + \frac{1}{2} (\ln t)^2 \right]_1^n$
 $= n + \frac{1}{2} \ln^2(n) - 1$

$n \geq 5 \Rightarrow \frac{1}{2} \ln^2(n) - 1 \geq 1$

d'où $n + \frac{1}{2} \ln^2(n) - 1 \geq n$

or $W_n \geq n + \frac{1}{2} \ln^2(n) - 1 \geq n$
 or $W_n \leq n \ln(n)$

$\forall n \geq 5; \boxed{n \leq W_n \leq n \ln(n)}$

c) $\lim_{n \rightarrow +\infty} (n) = +\infty \Rightarrow \boxed{\lim_{n \rightarrow +\infty} W_n = +\infty}$

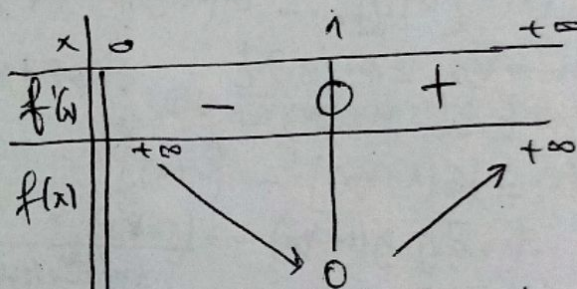
Problème 9 : BAC 2002 - Session Principale

$f(x) = x^2 - 2 \ln x - 1 \quad \mathcal{D}_f =]0, +\infty[$

1) a) $f \in \mathcal{C}^1$ sur $]0, +\infty[$

$f'(x) = 2x - \frac{2}{x} = \frac{2(x^2-1)}{x}$

$f'(x) = 0 \Leftrightarrow x^2 = 1 \Rightarrow x = 1$



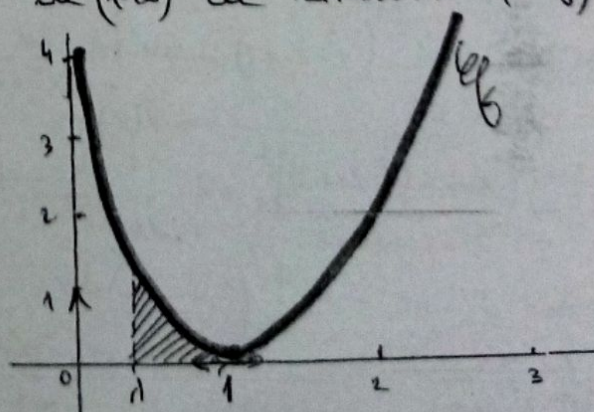
$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x^2 - 2 \ln x - 1) = +\infty$

alors $x=0$ Asymptote verticale

$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} (x^2 - 2 \ln x - 1)$
 $= \lim_{x \rightarrow +\infty} x^2 \left(1 - \frac{2 \ln x}{x^2} - \frac{1}{x^2} \right) = +\infty \quad (BI)$

$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \left(x - 2 \frac{\ln x}{x} - \frac{1}{x} \right) = +\infty$

$\Rightarrow f$ admet une BI au voisinage de $(+\infty)$ de direction $(0, 1)$



2) $\lambda \in]0, 1[$

a) $A(\lambda) = \int_{\lambda}^1 f(x) dx = \int_{\lambda}^1 (x^2 - 2\ln x - 1) dx$
 $= \left[\frac{1}{3}x^3 - 2(x\ln x - x) - x \right]_{\lambda}^1$
 $\Rightarrow A(\lambda) = \left(\frac{1}{3} + 2 - 1 \right) - \left(\frac{1}{3}\lambda^3 - 2\lambda\ln\lambda + \lambda \right)$
 $\Rightarrow A(\lambda) = \frac{4}{3} - \frac{1}{3}\lambda^3 + 2\lambda\ln(\lambda) - \lambda$ (ua)
 b) $\lim_{\lambda \rightarrow 0^+} \lambda\ln(\lambda) = 0 \Rightarrow \lim_{\lambda \rightarrow 0^+} A(\lambda) = \frac{4}{3}$

1) a) $n \geq 2, 1 \leq k \leq n-1$
 On $\frac{1}{n} f\left(\frac{k+1}{n}\right) \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \leq \frac{1}{n} f\left(\frac{k}{n}\right)$
 ma $1 \leq k \leq n-1 \Rightarrow \frac{1}{n} \leq \frac{k}{n} \leq \frac{n-1}{n} \leq 1$
 $\hookrightarrow 1 \leq k \leq n-1 \Rightarrow 2 \leq k+1 \leq n \Rightarrow \frac{k+1}{n} \leq 1$
 $\Rightarrow \left[\frac{k}{n}, \frac{k+1}{n} \right] \subset]0, 1[\hookrightarrow f \downarrow$
 d'où $\frac{k}{n} \leq x \leq \frac{k+1}{n}$
 $f \downarrow \text{ sur }]0, 1[\Rightarrow f\left(\frac{k+1}{n}\right) \leq f(x) \leq f\left(\frac{k}{n}\right)$
 comme $\frac{k}{n} < \frac{k+1}{n} \Rightarrow \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(\frac{k+1}{n}\right) dx \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(\frac{k}{n}\right) dx$
 $\Rightarrow f\left(\frac{k+1}{n}\right) \left[x \right]_{\frac{k}{n}}^{\frac{k+1}{n}} \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \leq f\left(\frac{k}{n}\right) \left[x \right]_{\frac{k}{n}}^{\frac{k+1}{n}}$
 $\Rightarrow f\left(\frac{k+1}{n}\right) \left[\frac{k+1}{n} - \frac{k}{n} \right] \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \leq f\left(\frac{k}{n}\right) \left[\frac{k+1}{n} - \frac{k}{n} \right]$
 $\Rightarrow \frac{1}{n} f\left(\frac{k+1}{n}\right) \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \leq \frac{1}{n} f\left(\frac{k}{n}\right)$

b) $\sum_{k=1}^{n-1} \frac{1}{n} f\left(\frac{k+1}{n}\right) \leq \sum_{k=1}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \leq \sum_{k=1}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right)$
 $\Rightarrow \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) \leq \int_{\frac{1}{n}}^1 f(x) dx \leq \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)$
 $\Rightarrow \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) \leq A\left(\frac{1}{n}\right) \leq \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)$

2) $S_n = \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)$

a) On $A\left(\frac{1}{n}\right) \leq S_n \leq \frac{1}{n} f\left(\frac{1}{n}\right) + A\left(\frac{1}{n}\right)$

ma $\sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) = \sum_{k=2}^n f\left(\frac{k}{n}\right) = \sum_{k=1}^n f\left(\frac{k}{n}\right) - f\left(\frac{1}{n}\right)$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) = \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) - \frac{1}{n} f\left(\frac{1}{n}\right)$$

$$\Rightarrow S_n = \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) - \frac{1}{n} f\left(\frac{1}{n}\right)$$

$$\text{a) } \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) \leq A\left(\frac{1}{n}\right)$$

$$\Rightarrow S_n + \frac{1}{n} f\left(\frac{1}{n}\right) \leq A\left(\frac{1}{n}\right)$$

$$\text{alors } S_n \leq A\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{1}{n}\right) \quad (1)$$

$$\text{d' } \sum_{k=2}^{n-1} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) - f\left(\frac{1}{n}\right)$$

$$\text{a) } f\left(\frac{1}{n}\right) = f(1) = 0$$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = S_n$$

$$\text{d' } \text{comme } A\left(\frac{1}{n}\right) \leq \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)$$

$$\Rightarrow A\left(\frac{1}{n}\right) \leq S_n \quad (2)$$

$$(1) \text{ et } (2) \Rightarrow A\left(\frac{1}{n}\right) \leq S_n \leq A\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{1}{n}\right)$$

b) $\lim_{n \rightarrow +\infty} A\left(\frac{1}{n}\right) = \lim_{\lambda \rightarrow 0^+} A(\lambda) = \frac{4}{3}$

$$\text{d' } \frac{1}{n} f\left(\frac{1}{n}\right) = \frac{1}{n} \left[\frac{1}{n^3} - 2\ln\left(\frac{1}{n}\right) - 1 \right]$$

$$= \frac{1}{n^3} + 2\frac{\ln n}{n} - \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \frac{1}{n} f\left(\frac{1}{n}\right) = \lim_{n \rightarrow +\infty} \left[\frac{1}{n^3} + 2\frac{\ln n}{n} - \frac{1}{n} \right] = 0$$

$\Rightarrow$ d'après le théorème de Gendarmes

on obtient : $\lim_{n \rightarrow +\infty} S_n = \frac{4}{3}$

3) a) On $\forall n \geq 2, \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$

• pour $n=2 \Rightarrow \sum_{k=1}^1 k^2 = 1^2 = \frac{2(2-1)(4-1)}{6} = 1$

• Supposons $\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$ vrai

• On $\sum_{k=1}^n k^2 = \frac{(n+1)(n)(2n+1)}{6} ?$

$$E_n = \sum_{k=1}^n k^2 = \sum_{k=1}^{n-1} k^2 + n^2$$

$$= \frac{n(n-1)(2n-1)}{6} + n^2$$

$$= \frac{n(n-1)(2n-1) + 6n^2}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

concl: $\forall n \geq 2, \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$

$$b) S_n = \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{k=1}^{n-1} \left(\frac{k^2}{n^2} - 2 \ln\left(\frac{k}{n}\right) - 1 \right)$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \frac{k^2}{n^2} - \frac{2}{n} \sum_{k=1}^{n-1} \ln\left(\frac{k}{n}\right) - \frac{1}{n} \sum_{k=1}^{n-1} 1$$

$$= \frac{1}{n} \cdot \frac{1}{n^2} \sum_{k=1}^{n-1} k^2 - \frac{2}{n} \left[\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) + \dots + \ln\left(\frac{n-1}{n}\right) \right] - \frac{1}{n} \cdot (n-1)$$

$$= \frac{1}{n^3} \cdot \frac{n(n-1)(2n-1)}{6} - \frac{2}{n} \ln\left(\frac{1 \times 2 \times \dots \times (n-1)}{n^{n-1}}\right) - 1 + \frac{1}{n}$$

$$= \frac{(n-1)(2n-1)}{6n^2} - \frac{2}{n} \ln\left(\frac{n!}{n^n}\right) - 1 + \frac{1}{n}$$

$$\Rightarrow S_n = \frac{(n-1)(2n-1)}{6n^2} + \frac{2}{n} \ln\left(\frac{n^n}{n!}\right) - 1 + \frac{1}{n}$$

$$= \frac{(n-1)(2n-1)}{6n^2} + \frac{2}{n} \ln\left(\frac{n}{n!}\right)^n - 1 + \frac{1}{n}$$

$$\Rightarrow S_n = \frac{(n-1)(2n-1)}{6n^2} + 2 \ln\left(\frac{n}{n!}\right) - 1 + \frac{1}{n}$$

c) Déduisons $\lim_{n \rightarrow \infty} \left(\frac{n}{\sqrt[n]{n!}} \right)$?

on pose $U_n = \frac{n}{\sqrt[n]{n!}}$

$$\Rightarrow S_n = \frac{(n-1)(2n-1)}{6n^2} + 2 \ln(U_n) - 1 + \frac{1}{n}$$

$$\Rightarrow 2 \ln(U_n) = S_n + 1 - \frac{1}{n} - \frac{2n^2 - 3n + 1}{6n^2}$$

$$\Rightarrow \ln(U_n) = \frac{1}{2} S_n + \frac{1}{2} - \frac{1}{2n} - \frac{2n^2 - 3n + 1}{12n^2}$$

$$\text{alors } U_n = e^{\frac{1}{2} S_n + \frac{1}{2} - \frac{1}{2n} - \frac{2n^2 - 3n + 1}{12n^2}}$$

$$\text{on } \lim S_n = \frac{4}{3} \quad \text{et } \lim_{n \rightarrow \infty} \frac{1}{2n} = 0 \quad \text{et } \lim_{n \rightarrow \infty} \frac{2n^2 - 3n + 1}{12n^2} = \frac{1}{6}$$

$$\Rightarrow \lim U_n = e^{\frac{1}{2}(\frac{4}{3}) + \frac{1}{2} - \frac{1}{6}} = e^1 = e$$

concl: $\lim_{n \rightarrow \infty} \left(\frac{n}{\sqrt[n]{n!}} \right) = e$

Problème 10 : BAC 2001 - Session Principale

I) $f(x) = \frac{2}{e^x + e^{-x}} \quad \mathcal{D}_f = \mathbb{R}$

a) $f \in \mathcal{C}^\infty$ sur $\mathbb{R}$ car $e^x + e^{-x} \neq 0$

$$f'(x) = \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{-2e^{-x}(e^{2x} - 1)}{(e^x + e^{-x})^2}$$

$$f'(x) = 0 \Leftrightarrow e^{2x} - 1 = 0 \Leftrightarrow e^{2x} = 1 \Leftrightarrow x = 0$$

x	$-\infty$	0	$+\infty$
f'(x)	-	0	+

$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{2}{e^x + e^{-x}} = 0$

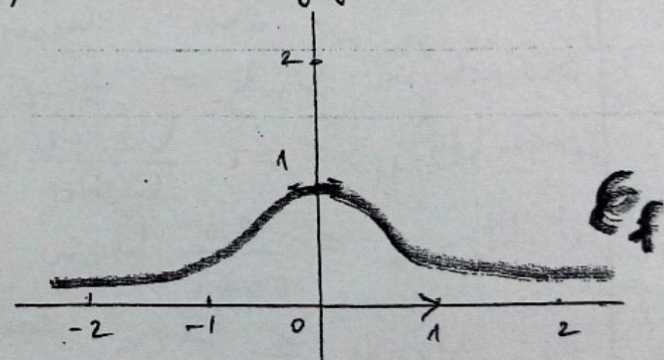
$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{2}{e^x + e^{-x}} = 0$

$f(0) = \frac{2}{1+1} = 1$

d'après le TV de f on a :

$$\forall x \in \mathbb{R}, \quad 0 < f(x) \leq 1$$

b) $y = 0$ Asymptote horizontale ($\neq 0$)



2) $g(x) = \ln(\tan x) \quad / x \in]0, \frac{\pi}{2}[$

a) on a : $x \mapsto \tan x \in]0, +\infty[\Rightarrow g \in]0, \frac{\pi}{2}[$

$$g'(x) = \frac{(\tan x)'}{\tan x} = \frac{1 + \tan^2 x}{\tan x}$$

b) $g'(x) > 0 \Rightarrow g$ strict $\nearrow$ sur $]0, \frac{\pi}{2}[$

$$\lim_{0^+} g(x) = \lim_{0^+} \ln(\tan x) = -\infty$$

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} \ln(\tan x) = +\infty$$

• $g \subseteq \text{strict } \nearrow \text{ sur }]0, \frac{\pi}{2}[\Rightarrow g \text{ réalise une bijection de }]0, \frac{\pi}{2}[\text{ sur } \mathbb{R}$

alors g admet une fonction réciproque $g^{-1} = h$ définie sur $\mathbb{R}$.

$$\bullet \text{ on pose } h(0) = x \Leftrightarrow g(x) = 0$$

$$\Leftrightarrow \ln(\tan x) = 0 \Leftrightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \text{ alors } \boxed{h(0) = \frac{\pi}{4}}$$

c) on a $g \subseteq \text{sur }]0, \frac{\pi}{2}[$ et $g'(x) \neq 0$

$$\Rightarrow h \subseteq \text{sur } \mathbb{R}$$

$$h'(x) = \frac{1}{g'(h(x))} = \frac{1}{1 + \tan^2(h(x))}$$

$$\Leftrightarrow h'(x) = \frac{\tan(h(x))}{1 + \tan^2(h(x))}$$

$$\text{or } g(h(x)) = x \Leftrightarrow \ln(\tan(h(x))) = x$$

$$\Leftrightarrow \tan(h(x)) = e^x \Rightarrow h'(x) = \frac{e^x}{1 + e^{2x}}$$

$$\Rightarrow h'(x) = \frac{e^x}{1 + e^{2x}} \cdot \frac{e^{-x}}{e^{-x}} = \frac{1}{e^x + e^{-x}} = \frac{1}{2} f(x)$$

$$\Leftrightarrow \boxed{2h'(x) = f(x)} \quad \forall x \in \mathbb{R}$$

$$\bullet \forall t \in \mathbb{R}; f(t) = 2h'(t)$$

$$\text{alors } \int_0^x f(t) dt = 2 \int_0^x h'(t) dt$$

$$\Rightarrow \int_0^x f(t) dt = 2[h(t)]_0^x = 2(h(x) - h(0))$$

$$\text{or } h(0) = \frac{\pi}{4} \Rightarrow \boxed{\int_0^x f(t) dt = 2h(x) - \frac{\pi}{2}}$$

$$\text{D) } F_n(x) = \int_0^x f(t) dt \quad / x \in [0, +\infty[$$

$$\text{N) } F_n(x) = \int_0^x f(t) dt = 2h(x) - \frac{\pi}{2}$$

$$\lim_{x \rightarrow +\infty} F_n(x) = \lim_{x \rightarrow +\infty} \left[2h(x) - \frac{\pi}{2} \right]$$

$$\text{or } \lim_{x \rightarrow +\infty} h(x) = \frac{\pi}{2} \text{ car } \lim_{x \rightarrow +\infty} g(x) = +\infty$$

$$\Rightarrow \lim_{x \rightarrow +\infty} F_n(x) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{2) } K(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} \quad D_K = \mathbb{R}$$

$$K \subseteq \text{sur } \mathbb{R} \text{ car } e^x + e^{-x} \neq 0$$

$$K'(t) = \frac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{(e^t + e^{-t})^2} = \frac{4}{(e^t + e^{-t})^2} = [f(t)]^2$$

$$\bullet F_2(x) = \int_0^x f^2(t) dt = \int_0^x K'(t) dt = [K(t)]_0^x = K(x) - K(0)$$

$$\text{or } K(0) = 0 \Rightarrow \boxed{F_2(x) = K(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$$

$$\bullet \lim_{x \rightarrow +\infty} F_2(x) = \lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow +\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

$$\Rightarrow \boxed{\lim_{x \rightarrow +\infty} F_2(x) = 1}$$

3a) F_n est la primitive sur $\mathbb{R}^+$ de $f''(t)$ qui s'annule en 0

$$\Rightarrow \text{on a } \begin{cases} F_n \subseteq \text{sur } [0, +\infty[\\ F_n'(x) = f''(x) \\ F_n(0) = 0 \end{cases}$$

d'où $F_n \subseteq \text{sur } [0, +\infty[\Rightarrow F_n \subseteq \text{sur } \mathbb{R}^+$

$$F_n'(x) = f''(x) \geq 0 \text{ car } 0 < f(x) \leq 1$$

$$\Rightarrow F_n([0, +\infty[) = [F_n(0), \lim_{x \rightarrow +\infty} F_n(x)]$$

$$\text{or } F_n(0) = 0 \Rightarrow \boxed{F_n([0, +\infty[) = [0, \lim_{x \rightarrow +\infty} F_n(x)]}$$

$$\text{b) } f(t) - 2e^{-t} = \frac{2}{e^t + e^{-t}} - 2e^{-t} = \frac{2 - 2 - 2e^{-2t}}{e^t + e^{-t}} = \frac{-2e^{-2t}}{e^t + e^{-t}} < 0$$

$$\Rightarrow \forall t \geq 0, f(t) < 2e^{-t}$$

• on a d'après I) 1/a) $0 < f(t) \leq 1$

$$\Rightarrow 0 < f''(t) \leq 1$$

$$\text{or } 0 < f(t) < 2e^{-t}$$

$$\text{Produit: } 0 < f''(t) \leq 2e^{-t}$$

donc $0 < \int_0^x f^n(t) dt \leq 2 \int_0^x e^{-t} dt$

$$\Rightarrow F_n(x) \leq 2 [-e^{-t}]_0^x$$

$$\Rightarrow F_n(x) \leq 2 - 2e^{-x} \leq 2$$

car: $\forall x \geq 0, F_n(x) \leq 2$

F_n est bornée sur $[0, +\infty[$ et majorée

par 2 $\Rightarrow \lim_{x \rightarrow +\infty} F_n(x) = l$ (finie)

$$c) f(t) - e^{-t} = \frac{2}{e^t + e^{-t}} - e^{-t} = \frac{2 - 1 - e^{-2t}}{e^t + e^{-t}}$$

$$= \frac{e^{-2t}(e^{2t} - 1)}{(e^t + e^{-t})} \geq 0 \quad \forall t \geq 0$$

alors $\forall t \geq 0, f(t) \geq e^{-t}$

donc $[f(t)]^n \geq e^{-nt}$

alors $\int_0^x f^n(t) dt \geq \int_0^x e^{-nt} dt$

$$\Rightarrow F_n(x) \geq -\frac{1}{n} [e^{-nt}]_0^x$$

$$\Rightarrow F_n(x) \geq \frac{1 - e^{-nx}}{n}$$

$$\lim_{x \rightarrow +\infty} \frac{1 - e^{-nx}}{n} = \frac{1}{n}$$

$$\lim_{x \rightarrow +\infty} F_n(x) \geq \frac{1}{n} \text{ alors } \lim_{x \rightarrow +\infty} F_n(x) \neq 0$$

$$3) \lim_{x \rightarrow +\infty} F_n(x) = U_n$$

$$a) U_1 = \lim_{x \rightarrow +\infty} F_1(x) = \frac{\pi}{2}$$

$$U_2 = \lim_{x \rightarrow +\infty} F_2(x) = 1$$

$$b) f^{n+1}(t) \cdot f'(t) \cdot k(t) = f^n(t) \cdot \frac{2(e^t - e^{-t})}{(e^t + e^{-t})^2} \cdot \frac{(e^t - e^{-t})}{e^t + e^{-t}}$$

$$= f^n(t) \cdot \frac{2}{e^t + e^{-t}} \cdot \frac{-(e^t - e^{-t})}{(e^t + e^{-t})^2}$$

$$= f^n(t) \left[\frac{4 - (e^t + e^{-t})^2}{(e^t + e^{-t})^2} \right] = f^n(t) [f^2(t) - 1]$$

$$\Leftrightarrow f^{n+1}(t) \cdot f'(t) \cdot k(t) = f^{n+2}(t) - f^n(t)$$

$$c) I = \int_0^x f^{n+1}(t) \cdot f'(t) \cdot k(t) dt$$

on pose $u(t) = k'(t) \rightarrow u'(t) = k''(t) = f(t)$
 $v(t) = f^n(t) \cdot f(t) \rightarrow v'(t) = \frac{1}{n} f^n(t)$

$$\Rightarrow I = \left[\frac{1}{n} f^n(t) \cdot k(t) \right]_0^x - \frac{1}{n} \int_0^x f^{n+2}(t) dt$$

$$= \frac{1}{n} f^n(x) \cdot k(x) - \frac{1}{n} \int_0^x f^{n+2}(t) dt$$

and: $\forall n \in \mathbb{N}^* \quad \forall x \geq 0$

$$\int_0^x f^{n+1}(t) \cdot f'(t) \cdot k(t) dt = \frac{1}{n} k(x) \cdot f^n(x) - \frac{1}{n} \int_0^x f^{n+2}(t) dt$$

d) on a $f^{n+1}(t) \cdot f'(t) \cdot k(t) = f^{n+2}(t) - f^n(t)$

$$\Rightarrow \int_0^x f^{n+1}(t) \cdot f'(t) \cdot k(t) dt = \int_0^x f^{n+2}(t) dt - \int_0^x f^n(t) dt$$

$$\Leftrightarrow \frac{1}{n} k(x) \cdot f^n(x) - \frac{1}{n} \int_0^x f^{n+2}(t) dt = \int_0^x f^{n+2}(t) dt - \int_0^x f^n(t) dt$$

$$\Leftrightarrow \frac{1}{n} k(x) \cdot f^n(x) - \frac{1}{n} F_{n+2}(x) = F_{n+2}(x) - F_n(x)$$

$$\Leftrightarrow k(x) \cdot f^n(x) - F_{n+2}(x) = n F_{n+2}(x) - n F_n(x)$$

$$\Rightarrow (n+1) F_{n+2}(x) - n F_n(x) = k(x) \cdot f^n(x)$$

$\lim_{x \rightarrow +\infty} F_n(x) = U_n$ et $\lim_{x \rightarrow +\infty} F_{n+2}(x) = U_{n+2}$

$\lim_{x \rightarrow +\infty} (n+1) = n+1$ et $\lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow +\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$

$\lim_{x \rightarrow +\infty} f(x) = 0 \Rightarrow \lim_{x \rightarrow +\infty} f^n(x) = 0^n = 0$

$$\Rightarrow (n+2) U_{n+2} - n U_n = 0 \Leftrightarrow U_{n+2} = \frac{n}{n+2} U_n$$

4) a) on a $\forall p \in \mathbb{N}^* \quad U_{p+2} = \frac{p}{p+2} U_p$

Par itération:

Pour $p=1 \rightarrow U_3 = \frac{1}{2} U_1$

$p=3 \rightarrow U_5 = \frac{3}{4} U_3$

$p=5 \rightarrow U_7 = \frac{5}{6} U_5$

$\vdots$

$p=2n-1 \rightarrow U_{2n+1} = \frac{2n-1}{2n} U_{2n-1}$

donc: $U_{2n+1} = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \dots \times \frac{2n-1}{2n} \cdot U_1$

on $U_1 = \frac{\pi}{2} \Rightarrow U_{2n+1} = \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2 \times 4 \times 6 \times \dots \times (2n)} \cdot \frac{\pi}{2}$

Écrivons maintenant les termes paires

$$\text{ma } \forall p \in \mathbb{N}^* \quad U_{p+2} = \frac{p}{p+1} U_p \Rightarrow$$

$$p=2 \longrightarrow U_4 = \frac{2}{3} U_2$$

$$p=4 \longrightarrow U_6 = \frac{4}{5} U_4$$

$$p=6 \longrightarrow U_8 = \frac{6}{7} U_6$$

$$\vdots$$

$$p=2n \longrightarrow U_{2n+2} = \frac{2n}{2n+1} U_{2n}$$

Produit : $U_{2n+2} = \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \dots \times \frac{2n}{2n+1} U_2$

or $U_2 = 1 \Rightarrow U_{2n+2} = \frac{2 \times 4 \times 6 \times \dots \times (2n)}{3 \times 5 \times 7 \times \dots \times (2n+1)}$

b) On a $U_n \downarrow$

ma $\forall k \geq 0, \quad 0 < f^{(k)}(t) \leq 1$

$f^{(n)}(t) \geq 0 \Rightarrow 0 < f^{(n+1)}(t) \leq f^{(n)}(t)$

pour $0 \leq x \Rightarrow 0 < \int_0^{x_{n+1}} f^{(n)}(t) dt \leq \int_0^{x_n} f^{(n)}(t) dt$

$\Rightarrow 0 < F_{n+1}(x) \leq F_n(x)$

alors F_n est strict $\downarrow$

c) On a $\forall n \in \mathbb{N} \quad \frac{2n}{2n+1} \leq \frac{U_{2n+2}}{U_{2n+1}} \leq 1$

ma de plus $U_n \downarrow \Rightarrow U_{2n+2} \leq U_{2n+1}$

comme $U_{2n+1} > 0 \Rightarrow \frac{U_{2n+2}}{U_{2n+1}} \leq 1$ ①

et $U_{2n+2} = \frac{2n}{2n+1} U_{2n}$

a. $U_{2n+1} \leq U_{2n}$ car $U_n \downarrow$

$\Rightarrow \frac{2n}{2n+1} U_{2n+1} \leq \frac{2n}{2n+1} U_{2n}$

donc $\frac{2n}{2n+1} U_{2n+1} \leq U_{2n+2}$

comme $U_{2n+1} > 0 \Rightarrow \frac{2n}{2n+1} \leq \frac{U_{2n+2}}{U_{2n+1}}$

Donc ② $\Rightarrow \frac{2n}{2n+1} \leq \frac{U_{2n+2}}{U_{2n+1}} \leq 1$

$\lim_{n \rightarrow \infty} \frac{2n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{2n}} = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{U_{2n+2}}{U_{2n+1}} = 1$

d) on pose : $V_n = \left[\frac{2 \times 4 \times 6 \times \dots \times (2n)}{3 \times 5 \times 7 \times \dots \times (2n+1)} \right]^2 \cdot \frac{2}{2n+1}$

Ma $\lim V_n = \pi$

ma $\frac{U_{2n+2}}{U_{2n+1}} = \frac{2 \times 4 \times 6 \times \dots \times (2n)}{3 \times 5 \times 7 \times \dots \times (2n+1)} \cdot \frac{2}{2n+1}$

$\Rightarrow \frac{U_{2n+2}}{U_{2n+1}} = \left[\frac{2 \times 4 \times 6 \times \dots \times (2n)}{3 \times 5 \times 7 \times \dots \times (2n+1)} \right]^2 \cdot \frac{1}{2n+1}$

$\Rightarrow \frac{U_{2n+2}}{U_{2n+1}} = \left[\frac{2 \times 4 \times 6 \times \dots \times (2n)}{3 \times 5 \times 7 \times \dots \times (2n+1)} \right]^2 \cdot \frac{2}{2n+1} \cdot \frac{1}{\pi}$

$\Leftrightarrow \frac{U_{2n+2}}{U_{2n+1}} = V_n \cdot \frac{1}{\pi} \Leftrightarrow \lim_{n \rightarrow \infty} \frac{U_{2n+2}}{U_{2n+1}} = 1$

$\Leftrightarrow \lim_{n \rightarrow \infty} \left(\frac{V_n}{\pi} \right) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} V_n = \pi$

Concl $\lim_{n \rightarrow \infty} \left[\frac{2 \times 4 \times 6 \times \dots \times (2n)}{3 \times 5 \times 7 \times \dots \times (2n+1)} \right]^2 \cdot \frac{2}{2n+1} = \pi$

Problème 11: BAC 2001 - Session de Contrôle

A 1) $g(x) = (1-x)e^{-x} + 1, \quad \text{Dg} = \mathbb{R}$

$g \in \mathcal{C}^1$ sur $\mathbb{R}$,

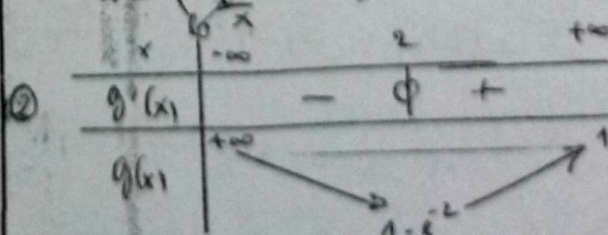
$g'(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$

on résout : $g'(x) = 0 \Leftrightarrow x = 2$

$\lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow -\infty} (1-x)e^{-x} = +\infty$

$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} [(1-x)e^{-x} + 1] = \lim_{x \rightarrow +\infty} \left(\frac{1-x}{e^x} + 1 \right)$

$= \lim_{x \rightarrow +\infty} \left(\frac{1}{e^x} - \frac{x}{e^x} + 1 \right) = 1, \quad g(2) = 1 \cdot e^{-2}$



$1 - e^{-2}$ minimum absolu pour $g(x)$
 $\Rightarrow \forall x \in \mathbb{R}, g(x) \geq 1 - e^{-2} > 0 \Rightarrow g(x) > 0$

2) $f(x) = x e^{-x} + x$, $\mathcal{D}_f = \mathbb{R}$

a) $f \in \mathcal{C}^\infty$ sur $\mathbb{R}$

$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \left(\frac{x}{e^x} + x \right) = \lim_{x \rightarrow +\infty} \left(\frac{1}{e^x} + x \right) = +\infty$

$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} (x e^{-x} + x) = -\infty$

$f'(x) = e^{-x} - x e^{-x} + 1 = (1-x) e^{-x} + 1 = g(x)$

$\Rightarrow \forall x \in \mathbb{R}, f'(x) > 0$

x	$-\infty$	$+\infty$
$f'(x)$		+
$f(x)$	$-\infty$	$+\infty$

b) $\lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{x \rightarrow +\infty} x e^{-x} = \lim_{x \rightarrow +\infty} \frac{1}{e^x} = 0$

$\Rightarrow \Delta: y = x$ Asymptote oblique en $(+\infty)$.

c) $f(x) - x = x e^{-x}$

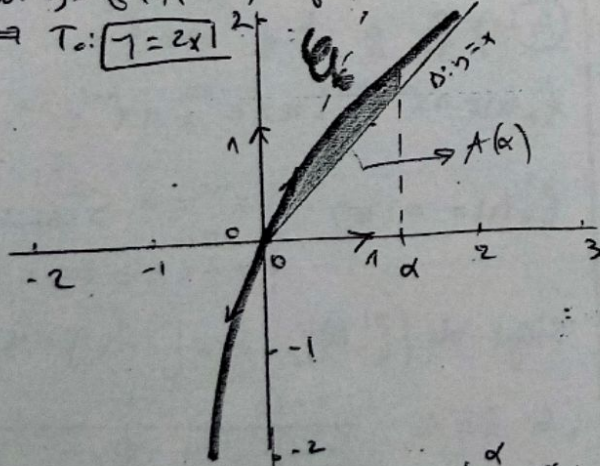
x	$-\infty$	0	$+\infty$
$f(x) - x$		0	+
Positivité	Écart des courbes	Écart des courbes	Écart des courbes

d) BI en $(-\infty)$: $\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} \frac{x e^{-x} + x}{x} = \lim_{x \rightarrow -\infty} (e^{-x} + 1) = +\infty$

$\Rightarrow$ Écart des BI en $(-\infty)$ de direction $(0, 1)$

T₀: $y = f'(0)(x-0) + f(0)$ or $f'(0) = 2$

$\Rightarrow T_0: y = 2x$



3) a) $A(x) = \int_0^x (f(t) - t) dt = \int_0^x t e^{-t} dt$

on pose $u(t) = t \rightarrow u'(t) = 1$

$v'(t) = e^{-t} \rightarrow v(t) = -e^{-t}$

$\Rightarrow A(x) = [-t e^{-t}]_0^x + \int_0^x e^{-t} dt = [-t e^{-t} - e^{-t}]_0^x$

$\Rightarrow A(x) = (-x e^{-x} - e^{-x}) - (0 - 1) = 1 - (x+1) e^{-x}$

b) $A(1) = 1 - 2e^{-1}$

$\lim_{x \rightarrow +\infty} A(x) = \lim_{x \rightarrow +\infty} 1 - (1+x) e^{-x} = 1 + \lim_{x \rightarrow +\infty} \left(-\frac{1}{e^x} - \frac{x}{e^x} \right) = 1$

$\Rightarrow \lim_{x \rightarrow +\infty} A(x) = 1$

3) $F_n(x) = \int_0^x t^n e^{-t} dt$, $\mathcal{D}_F = [0, +\infty)$

1) $F_1(x) = \int_0^x t e^{-t} dt = A(1) = 1 - (1+x) e^{-x}$

2) a) $F_n(x) = \int_0^x t^n e^{-t} dt$

on pose $u(t) = t^n \rightarrow u'(t) = n t^{n-1}$

$v'(t) = e^{-t} \rightarrow v(t) = -e^{-t}$

$\Rightarrow F_n(x) = [-t^n e^{-t}]_0^x + n \int_0^x t^{n-1} e^{-t} dt$

$\Rightarrow F_n(x) = -x^n e^{-x} + n F_{n-1}(x)$ $\forall n \geq 2$

b) Or $\forall n \in \mathbb{N}^*$ $F_n(x) = n! \left[1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x} \right]$

$\lim_{n \rightarrow 1} F_1(x) = 1 - \frac{x^0}{0!} e^{-x} - \frac{x^1}{1!} e^{-x} = 1 - (1+x) e^{-x}$ vérifie

Supposons que $F_n(x) = n! \left[1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x} \right]$ vrai

Deq: $F_{n+1}(x) = (n+1)! \left[1 - \sum_{k=0}^{n+1} \frac{x^k}{k!} e^{-x} \right]$

en effet: $F_{n+1}(x) = -x^{n+1} e^{-x} + (n+1) F_n(x)$

$\Rightarrow F_{n+1}(x) = -x^{n+1} e^{-x} + (n+1) \cdot n! \left[1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x} \right]$

$= -x^{n+1} e^{-x} + (n+1)! \left[1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x} \right]$

$\Rightarrow F_{n+1}(x) = (n+1)! \left[1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x} - \frac{x^{n+1}}{(n+1)!} e^{-x} \right]$

$\Rightarrow F_{n+1}(x) = (n+1)! \left[1 - \sum_{k=0}^{n+1} \frac{x^k}{k!} e^{-x} \right]$

Cml: $\forall n \in \mathbb{N}^*$

$F_n(x) = n! \left[1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x} \right]$

c) $\lim_{x \rightarrow +\infty} \frac{x^k e^{-x}}{k!} = \lim_{x \rightarrow +\infty} \frac{1}{k!} \frac{x^k}{e^x}$

$= \frac{1}{k!} \lim_{x \rightarrow +\infty} \frac{1}{e^x} = \frac{1}{k!} \cdot 0 = 0$ (k : constante)

$\Rightarrow \lim_{x \rightarrow +\infty} \sum_{k=0}^n \frac{x^k}{k!} e^{-x} = 0 \Rightarrow \lim_{x \rightarrow +\infty} F_n(x) = n!$

$$3) \text{ car } 0 \leq t \leq x \Rightarrow -x \leq -t \leq 0$$

$$e^1 \Rightarrow e^{-x} \leq e^{-t} \leq 1 \Rightarrow$$

$$a) t \geq 0 \Rightarrow t^n e^{-x} \leq t^n e^{-t} \leq t^n$$

$$\text{car } 0 < x \Rightarrow \int_0^x t^n e^{-x} dt \leq \int_0^x t^n e^{-t} dt \leq \int_0^x t^n dt$$

$$\Rightarrow e^{-x} \left[\frac{t^{n+1}}{n+1} \right]_0^x \leq f_n(x) \leq \left[\frac{t^{n+1}}{n+1} \right]_0^x$$

$$\Rightarrow \frac{x^{n+1}}{n+1} e^{-x} \leq f_n(x) \leq \frac{x^{n+1}}{n+1}$$

$$4) a) \frac{x^n}{n!} - \frac{1}{2} \frac{x^{n-1}}{(n-1)!} = \frac{2x^n - nx^{n-1}}{2n!}$$

$$= \frac{x^{n-1}(2x - n)}{2n!} \leq 0 \text{ car } x^n \geq 0$$

$$2x \leq n$$

$$\Rightarrow \frac{x^n}{n!} \leq \frac{1}{2} \frac{x^{n-1}}{(n-1)!} \quad \forall n \geq 2n$$

b) st p: un entier naturel: $2x \leq n_0 \leq p \leq n$

$$\text{car } \forall p \geq n_0, \frac{x^p}{p!} \leq \frac{1}{2} \frac{x^{p-1}}{(p-1)!}$$

par induction: $p = n_0 \Rightarrow \frac{x^{n_0}}{n_0!} \leq \frac{1}{2} \frac{x^{n_0-1}}{(n_0-1)!}$

$$p = n_0 + 1 \Rightarrow \frac{x^{n_0+1}}{(n_0+1)!} \leq \frac{1}{2} \frac{x^{n_0}}{n_0!}$$

$$\vdots$$

$$p = n \Rightarrow \frac{x^n}{n!} \leq \frac{1}{2} \frac{x^{n-1}}{(n-1)!}$$

Produit: $\frac{x^n}{n!} \leq \left(\frac{1}{2}\right)^{n-n_0+1} \frac{x^{n_0-1}}{(n_0-1)!}$

$$\Rightarrow \forall n \geq 2x, 0 < \frac{x^n}{n!} \leq \left(\frac{1}{2}\right)^{n-n_0+1} \frac{x^{n_0-1}}{(n_0-1)!}$$

$$\text{a) } n_0: \text{cté} \Rightarrow \frac{1}{2} \in]-1, 1[\Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{1}{2}\right)^{n-n_0+1} = 0$$

$$\text{cté } \lim_{n \rightarrow +\infty} \frac{x^{n_0-1}}{(n_0-1)!} = \frac{x^{n_0-1}}{(n_0-1)!} : \text{cté}$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{1}{2}\right)^{n-n_0+1} \frac{x^{n_0-1}}{(n_0-1)!} = 0 \Rightarrow \lim_{n \rightarrow +\infty} \frac{x^n}{n!} = 0$$

c) on a $f_n(x) = n! \left(1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x}\right)$

$$\text{car } \lim_{x \rightarrow +\infty} f_n(x) = n! \Rightarrow \lim_{x \rightarrow +\infty} \left(1 - \sum_{k=0}^n \frac{x^k}{k!} e^{-x}\right) = 0$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \sum_{k=0}^n \frac{x^k}{k!} e^{-x} = 1$$

$$\Rightarrow \left(\lim_{n \rightarrow +\infty} e^x \sum_{k=0}^n \frac{x^k}{k!} = 1 \right) e^x$$

$$\Rightarrow e^x = \lim_{n \rightarrow +\infty} \sum_{k=0}^n \frac{x^k}{k!}$$

5) a) on a $\frac{x^{n+1}}{(n+1)!} e^{-x} \leq f_{n+1}(x) \leq \frac{x^{n+1}}{(n+1)!}$

$$\text{par } x=1 \Rightarrow \frac{e^{-1}}{(n+1)!} \leq n! \left(1 - \sum_{k=0}^n \frac{1}{k!} e^{-1}\right) \leq \frac{1}{(n+1)!}$$

$$\Rightarrow \left(\frac{e^{-1}}{(n+1)!} \cdot \frac{1}{n!} \leq 1 - e^{-1} \sum_{k=0}^n \frac{1}{k!} \leq \frac{1}{(n+1)!} \cdot \frac{1}{n!} \right) e$$

$$\Rightarrow \frac{1}{(n+1)!} \leq e - \sum_{k=0}^n \frac{1}{k!} \leq \frac{e}{(n+1)!}$$

$$\Rightarrow \frac{1}{(n+1)!} + \sum_{k=0}^n \frac{1}{k!} \leq e \leq \sum_{k=0}^n \frac{1}{k!} + \frac{e}{(n+1)!}$$

$$\Rightarrow \sum_{k=0}^{n+1} \frac{1}{k!} \leq e \leq \sum_{k=0}^n \frac{1}{k!} + \frac{1}{n!} \text{ car } \frac{e}{n!} < \frac{1}{n!}$$

b) $d = \frac{1}{n!} < 10^{-2} \Rightarrow n! > 100 \Rightarrow n=5$

$$\text{car } 5! = 120 \Rightarrow \sum_{k=0}^5 \frac{1}{k!} < e < \sum_{k=0}^5 \frac{1}{k!} + \frac{1}{5!}$$

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \leq e \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{2}{5!}$$

Problème 12 : BAC 2000 - Session Principale

I) 1) $g(x) = x \ln x - x + 1 \quad \text{D}_g =]1, +\infty[$

a) $g \in \mathcal{C}^\infty \text{ sur }]1, +\infty[$

$$g'(x) = \ln x + x \left(\frac{1}{x}\right) - 1 = \ln x \geq 0$$

x	1	$+\infty$
$g'(x)$	0	+
$g(x)$	0	$+\infty$

$$g(1) = 0$$

$$\lim_{x \rightarrow +\infty} g = \lim_{x \rightarrow +\infty} x(\ln x - 1) + \frac{1}{x} = +\infty$$

b) on a $g([1, +\infty[) =]0, +\infty[$

$$\Rightarrow \forall x \geq 1, g(x) \geq 0$$

2) $f(x) = \frac{x-1}{\ln x} \quad \text{si } x > 1$

$$f(1) = 1$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x-1}{\ln x} = \lim_{x \rightarrow 1} \frac{1}{\frac{1}{x-1}} = \frac{1}{1} = 1$$

$$= f(1) \Rightarrow f \text{ est en } 1$$

$$\begin{aligned}
 3) a) (t-1) - (t-1)^2 - (1-\frac{1}{t}) &= (t-1) - (t-1)^2 - (\frac{t-1}{t}) \\
 &= (t-1) \left[1 - (t-1) - \frac{1}{t} \right] = (t-1) \left[\frac{2t-t^2-1}{t} \right] \\
 &= (t-1) \left[-\frac{(t-1)^2}{t} \right] = -\frac{(t-1)^3}{t} \leq 0 \quad \forall t \geq 1 \\
 \cdot (1-\frac{1}{t}) - (t-1) &= (\frac{t-1}{t}) - (t-1) = (\frac{t-1}{t})(1-t) \\
 &= -\frac{(t-1)^2}{t} \leq 0 \quad (2)
 \end{aligned}$$

$$1) d) (t) \rightarrow (t-1) - (t-1)^2 \leq 1 - \frac{1}{t} \leq t-1$$

$$\begin{aligned}
 b) \text{ pour } 1 \leq x \Rightarrow \int_1^x (t-1) - (t-1)^2 dt &\leq \int_1^x (1-\frac{1}{t}) dt \leq \int_1^x (t-1) dt \\
 \Rightarrow \left[\frac{(t-1)^2}{2} - \frac{(t-1)^3}{3} \right]_1^x &\leq \left[t - \ln t \right]_1^x \leq \left[\frac{(t-1)^2}{2} \right]_1^x \\
 \Rightarrow \left[\frac{(x-1)^2}{2} - \frac{(x-1)^3}{3} \right] &\leq x-1-\ln x \leq \frac{(x-1)^2}{2}
 \end{aligned}$$

$$c) \text{ pour } (x-1)^2 > 0 \Rightarrow \frac{1}{2} - \frac{x-1}{3} \leq \frac{x-1-\ln x}{(x-1)^2} \leq \frac{1}{2}$$

$$\begin{aligned}
 d) \text{ d'après } f \text{ en } 1^+ : \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1} &= \lim_{x \rightarrow 1^+} \frac{\frac{x-1}{\ln x} - 1}{x-1} \\
 &= \lim_{x \rightarrow 1^+} \frac{x-1-\ln x}{(x-1) \cdot \ln x} = \lim_{x \rightarrow 1^+} \frac{\frac{x-1-\ln x}{(x-1)^2}}{\left(\frac{\ln x}{x-1}\right)}
 \end{aligned}$$

$$\text{or } \lim_{x \rightarrow 1^+} \frac{\ln x}{x-1} = 1 \text{ et } \lim_{x \rightarrow 1^+} \frac{x-1-\ln x}{(x-1)^2} = \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1} = \frac{1}{2} : \text{(fini)} \text{ alors } f \text{ est dérivable en } 1^+ \text{ et } f'(1) = \frac{1}{2}$$

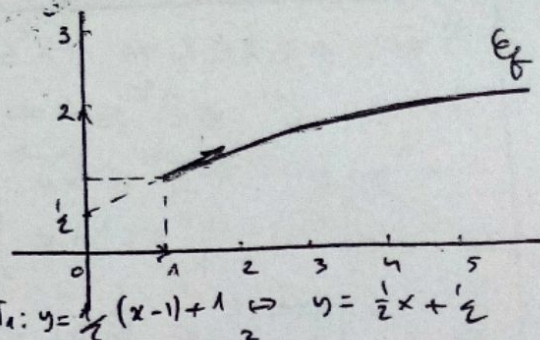
$$4) a) f \text{ est sur }]1, +\infty[\cup \{1\} =]1, +\infty[$$

$$\begin{aligned}
 \forall x > 1, f'(x) &= \left(\frac{x-1}{\ln x} \right)' = \frac{\ln x - (x-1)\frac{1}{x}}{(\ln x)^2} \\
 \Rightarrow f'(x) &= \frac{\ln x - 1 + \frac{1}{x}}{(\ln x)^2} = \frac{x \ln x - x + 1}{(\ln x)^2 \cdot x} = \frac{g(x)}{x f(x)} \\
 \Rightarrow f'(x) > 0 \quad \forall x > 1
 \end{aligned}$$

x	1	+	+	+
f'(x)		+		
f(x)	1		+	+

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} f(x) &= \lim_{x \rightarrow +\infty} \frac{x-1}{\ln x} \\
 &= \lim_{x \rightarrow +\infty} \frac{1-\frac{1}{x}}{\frac{\ln x}{x}} = \frac{1}{0^+} = +\infty
 \end{aligned}$$

$$\begin{aligned}
 b) \text{ BI: } \lim_{x \rightarrow +\infty} \frac{f(x)}{x} &= \lim_{x \rightarrow +\infty} \frac{\frac{x-1}{\ln x}}{x} = \lim_{x \rightarrow +\infty} \frac{1-\frac{1}{x}}{\ln x} = 0 \\
 \Rightarrow f &\text{ admet une BI en } (+\infty) \text{ de direction } (0)
 \end{aligned}$$



$$\begin{aligned}
 T_1: y &= \frac{1}{2}(x-1) + 1 \Leftrightarrow y = \frac{1}{2}x + \frac{1}{2} \\
 \text{II) } F(x) &= \int_x^{x^2} \frac{dt}{\ln t} \quad \text{si } x > 1 \\
 F(1) &= \ln 2
 \end{aligned}$$

$$\begin{aligned}
 1) a) \text{ or } x \leq t \leq x^2 \quad \text{car } x > 1 \\
 \ln 1 \Rightarrow \ln x \leq \ln t \leq \ln(x^2) \\
 \Rightarrow \frac{1}{\ln(x^2)} \leq \frac{1}{\ln t} \leq \frac{1}{\ln x} \\
 \text{d'où } \int_x^{x^2} \frac{1}{\ln t} dt \leq \int_x^{x^2} \frac{1}{\ln t} dt \leq \int_x^{x^2} \frac{1}{\ln x} dt \\
 \Rightarrow \frac{1}{\ln(x^2)} (t)_x^{x^2} \leq F(x) \leq \frac{1}{\ln x} (t)_x^{x^2} \\
 \Rightarrow \frac{x^2-x}{\ln(x^2)} \leq F(x) \leq \frac{x^2-x}{\ln x} \quad \forall x > 1
 \end{aligned}$$

$$b) \lim_{x \rightarrow +\infty} \frac{x^2-x}{\ln(x^2)} = \lim_{x \rightarrow +\infty} \frac{1-\frac{1}{x}}{\frac{\ln(x^2)}{x^2}} = \frac{1}{0^+} = +\infty$$

$$\begin{aligned}
 \Rightarrow \lim_{x \rightarrow +\infty} F(x) &= +\infty \\
 \frac{x-1}{\ln(x^2)} &\leq \frac{F(x)}{x} \leq \frac{x-1}{\ln x}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} \frac{x-1}{\ln(x^2)} &= \lim_{x \rightarrow +\infty} \frac{1-\frac{1}{x}}{\frac{2 \ln x}{x}} = +\infty \\
 \Rightarrow \lim_{x \rightarrow +\infty} \frac{F(x)}{x} &= +\infty \text{ alors la courbe } \\
 &\text{C' de } f \text{ admet une BI en } (+\infty) \text{ de direction } (0)
 \end{aligned}$$

$$2) a) x \leq t \leq x^2 \quad \forall x > 1$$

$$\begin{aligned}
 t \ln t > 0 \Rightarrow \frac{x}{t \ln t} \leq \frac{1}{t \ln t} \leq \frac{x^2}{t \ln t} \\
 \Rightarrow \frac{x}{t \ln t} \leq \frac{1}{t \ln t} \leq \frac{x^2}{t \ln t} \\
 b) \text{ pour } x \leq x^2 \Rightarrow \int_x^{x^2} \frac{x}{t \ln t} dt \leq \int_x^{x^2} \frac{1}{t \ln t} dt \leq \int_x^{x^2} \frac{x^2}{t \ln t} dt \\
 \Rightarrow x \int_x^{x^2} \frac{1}{t \ln t} dt \leq F(x) \leq x^2 \int_x^{x^2} \frac{1}{t \ln t} dt \\
 \Rightarrow x \left[\ln(\ln t) \right]_x^{x^2} \leq F(x) \leq x^2 \left[\ln(\ln t) \right]_x^{x^2}
 \end{aligned}$$

$$\Rightarrow x \ln(2) \leq f(x) \leq x^2 \ln(2)$$

$$c) \lim_{x \rightarrow 1^+} x \ln(2) = \ln 2 \quad \left\{ \begin{array}{l} \Rightarrow \lim_{x \rightarrow 1^+} F(x) = \ln 2 = f(1) \\ \lim_{x \rightarrow 1^+} x^2 \ln(2) = \ln 2 \Rightarrow f \text{ est en } 1^+ \end{array} \right.$$

$$3) a) \text{ on pose } h(t) = \frac{1}{\ln t} \in]1, +\infty[$$

$\Rightarrow h$ admet une primitive $H \in]1, +\infty[$ et telle que $H'(t) = h(t)$

$$\Rightarrow F(x) = \left[H(t) \right]_x^{x^2} = H(x^2) - H(x)$$

$$\text{comme } H \in]1, +\infty[\Rightarrow F \in]1, +\infty[$$

$$F'(x) = [H(x^2)]' - [H(x)]' \\ = 2x \cdot H'(x^2) - H'(x) = 2x \cdot h(x^2) - h(x) \\ = 2x \cdot \frac{1}{\ln(x^2)} - \frac{1}{\ln x} = \frac{2x}{2 \ln x} - \frac{1}{\ln x} = \frac{x-1}{\ln x}$$

$$\Rightarrow F'(x) = f(x) \quad \forall x > 1$$

$$b) \text{ on a } F \in]1, x] \text{ et } F \in]1, x]$$

$\Rightarrow$ d'après le TAF, $F \in]1, x]$

$$\Rightarrow \text{il existe } c \in]1, x[\text{ / } F'(c) = \frac{F(x) - F(1)}{x - 1}$$

$$\Rightarrow F(x) - F(1) = (x-1) \cdot F'(c)$$

$$c) \text{ d de } F \text{ en } 1^+ : \lim_{x \rightarrow 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(x-1)F'(c)}{x-1} = \lim_{x \rightarrow 1^+} F'(c)$$

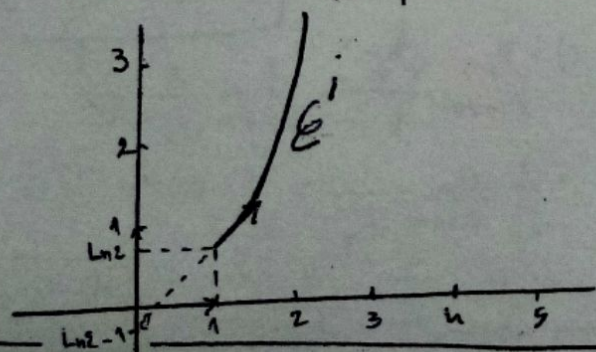
$$= \lim_{x \rightarrow 1^+} F'(c) = \lim_{c \rightarrow 1^+} f(c) = f(1) = 1$$

$$\Rightarrow f \text{ est en } 1^+ \text{ et } f'(1) = 1$$

$$4) \forall x > 1, f'(x) > 0$$

$$\begin{array}{c|c} x & 1 \quad \quad \quad +\infty \\ \hline F' & 1 \quad \quad \quad + \\ \hline f(x) & \end{array} \quad \lim_{x \rightarrow +\infty} F(x) = +\infty$$

Tc: $y = F'(1)(x-1) + F(1)$
 $\Rightarrow y = x-1 + \ln 2$



III

$$1) \text{ on a } F(x) = \int_1^x f(t) dt + \ln 2$$

$$\text{on pose } \varphi(x) = \int_1^x f(t) dt, \quad \text{sur }]1, +\infty[$$

φ est la primitive de f sur $]1, +\infty[$ qui s'annule en 1

$$\Rightarrow \begin{cases} \varphi \in]1, +\infty[\\ \varphi'(x) = f(x) \\ \varphi(1) = 0 \end{cases}$$

$$\text{on } F'(x) = f(x) \Rightarrow F'(x) = \varphi'(x)$$

$$\Rightarrow F(x) = \varphi(x) + k$$

$$\text{on } F(1) = \ln 2 \quad \left\{ \begin{array}{l} \Rightarrow k = \ln 2 \\ \varphi(1) = 0 \end{array} \right. \Rightarrow F(x) = \varphi(x) + \ln 2$$

$$\text{On a : } F(x) = \int_1^x f(t) dt + \ln 2$$

$$2) A(\alpha) = \int_1^\alpha f(t) dt = F(\alpha) - \ln 2$$

$$\text{on } \frac{\alpha^2 - \alpha}{\ln(\alpha^2)} \leq F(\alpha) \leq \frac{\alpha^2 - \alpha}{\ln \alpha}$$

$$\Rightarrow \frac{\alpha-1}{\ln(\alpha^2)} - \frac{\ln 2}{\alpha} \leq \frac{A(\alpha)}{\alpha} \leq \frac{\alpha-1}{\ln \alpha} - \frac{\ln 2}{\alpha}$$

$$\Rightarrow \lim_{\alpha \rightarrow +\infty} \frac{A(\alpha)}{\alpha} = 0 \quad \text{et} \quad \frac{1-\frac{1}{\alpha}}{\ln(\alpha^2)} - \frac{\ln 2}{\alpha^2} \leq \frac{A(\alpha)}{\alpha^2} \leq \frac{1-\frac{1}{\alpha}}{\ln \alpha} - \frac{\ln 2}{\alpha^2}$$

$$\Rightarrow \lim_{\alpha \rightarrow +\infty} \frac{A(\alpha)}{\alpha^2} = 0$$

Problème 13 : BAC 1999 - Session Principale

$$f_n(x) = x^n e^{-nx} \quad \text{Defn} = \mathbb{R}_+ \text{ et } n \in \mathbb{N}^*$$

$$(A) 1) f_n \in \mathcal{C}^1 \text{ sur } \mathbb{R}_+$$

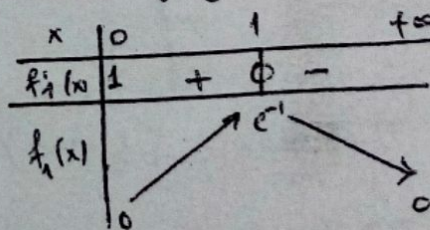
$$f'_n(x) = nx^{n-1} e^{-nx} - nx^n e^{-nx} = nx^{n-1} e^{-nx} (1-x)$$

$$f'_n(x) = 0 \Leftrightarrow \begin{cases} x^{n-1} = 0 \text{ ou } 1-x = 0 \end{cases} \begin{array}{l} x=0 \text{ si } n \geq 2 \\ \text{Absolu si } n=1 \end{array}$$

$$\text{Signe de } [f'_n(x)] = \text{sign}(1-x) \text{ sur } \mathbb{R}_+$$

$$\bullet \text{ Si } n=1$$

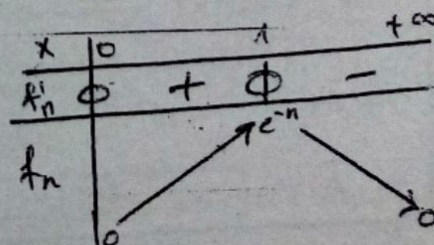
$$f'_1(0) = 1$$



$$\bullet \text{ Si } n \geq 2$$

$$f_n(0) = 0$$

$$f_n(1) = e^{-n}$$



$$\lim_{t \rightarrow 0} f_n(t) = \lim_{t \rightarrow 0} x^n e^{-nx} = \lim_{t \rightarrow 0} \frac{x^n}{e^{nx}} = \lim_{t \rightarrow 0} \left(\frac{x}{e^x} \right)^n$$

b) Position de E_n et E_{n+1} :

$$\text{on pose } \varphi(x) = f_{n+1}(x) - f_n(x) = x^{n+1} e^{-(n+1)x} - x^n e^{-nx}$$

$$\Rightarrow \varphi(x) = x^n e^{-nx} (x e^{-x} - 1) = f_n(x) \cdot [f_1(x) - 1]$$

$$\text{or } \forall x \geq 0, f_n(x) \geq 0 \text{ et } f_1(x) \leq e^{-1}$$

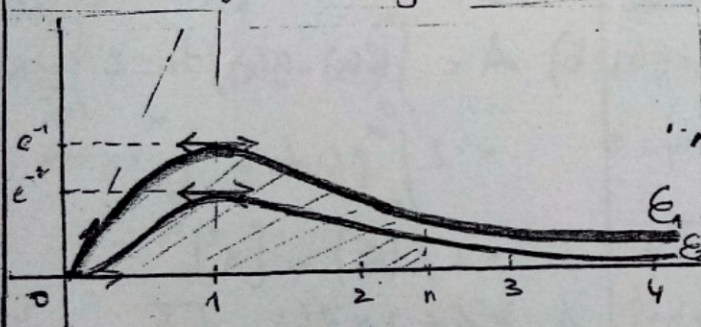
$$\Rightarrow f_1(x) - 1 < 0 \Rightarrow \varphi(x) \leq 0 \quad \forall x \in \mathbb{R}$$

d'où E_{n+1} est toujours située au dessous de $E_n \quad \forall x \geq 0$

2a) si $n=1 \Rightarrow E_1$ admet en 0 une tangente d'éq $y = f_1'(0)(x-0) + f_1(0)$

$$\Rightarrow \boxed{y=x}$$

Si $n \geq 2 \Rightarrow E_n$ admet à droite de 0 (pp) une demi-tangente horizontale.



$$b) S_n = \int_0^n f_1(x) dx = \int_0^n x e^{-x} dx$$

$$\text{on pose } \begin{cases} u(x) = x & \rightarrow u'(x) = 1 \\ v'(x) = e^{-x} & \rightarrow v(x) = -e^{-x} \end{cases}$$

$$S_n = \left[-x e^{-x} \right]_0^n + \int_0^n e^{-x} dx = \left[-x e^{-x} - e^{-x} \right]_0^n$$

$$\Rightarrow S_n = \left[-(1+x) e^{-x} \right]_0^n = -(1+n) e^{-n} + 1$$

$$\Rightarrow \boxed{S_n = 1 - (1+n) e^{-n}} \quad (ua)$$

$$c) \lim_{n \rightarrow +\infty} S_n = \lim_{n \rightarrow +\infty} (1 - (1+n) e^{-n})$$

$$= \lim_{n \rightarrow +\infty} 1 - \frac{n}{e^n} = 1 \Rightarrow \lim_{n \rightarrow +\infty} S_n = 1$$

$$\textcircled{B} F_n(x) = \int_0^x f_n(t) dt \quad / x \geq 0$$

$$1) \text{ Ma } \forall t \geq 0; \quad 0 \leq f_1(t) e^{\frac{1}{2}t} \leq 1$$

$$\text{ma } \forall x \geq 0 \quad 0 \leq f_1(x) \leq e^{-1}$$

$$\Rightarrow 0 \leq f_1(t) \leq e^{-1} \Leftrightarrow 0 \leq \frac{1}{2} t e^{-\frac{1}{2}t} \leq \frac{1}{e}$$

$$\Rightarrow 0 \leq t e^{-\frac{1}{2}t} \leq \frac{2}{e} \leq 1$$

$$\Rightarrow \forall t \geq 0, \quad \boxed{0 \leq f_1(t) \cdot e^{\frac{1}{2}t} \leq 1}$$

$$2) \text{ Ma } \forall t \geq 0; \quad 0 \leq f_n(t) \leq e^{-\frac{1}{2}t}$$

$$\text{ma } 0 \leq f_1(t) e^{\frac{1}{2}t} \leq 1$$

$$e^{\frac{1}{2}t} > 0 \Rightarrow 0 \leq f_1(t) \leq e^{-\frac{1}{2}t}$$

or $\forall n \in \mathbb{N}^*$ ma E_n est située au dessous de $E_1 \Rightarrow 0 \leq f_n(t) \leq f_1(t) \leq e^{-\frac{1}{2}t}$

$$\Rightarrow \forall t \geq 0, \quad 0 \leq f_n(t) \leq e^{-\frac{1}{2}t}$$

$$b) \text{ pu } 0 \leq x \Rightarrow 0 \leq \int_0^x f_n(t) dt \leq \int_0^x e^{-\frac{1}{2}t} dt$$

$$\Rightarrow 0 \leq f_n(x) \leq \left[-2e^{-\frac{1}{2}t} \right]_0^x$$

$$\Rightarrow 0 \leq f_n(x) \leq 2 - 2e^{-\frac{1}{2}x} \leq 2$$

$$\text{donc } \forall x \geq 0, \quad \boxed{0 \leq F_n(x) \leq 2}$$

$$\textcircled{C} \quad \forall x \geq 0, \quad G_n(u) = \int_0^u t^n e^{-t} dt$$

$$1) \text{ Ma } \forall n \geq 2, \quad G_n(u) = -u^n e^{-u} + n G_{n-1}(u)$$

$$\text{on pose } \begin{cases} A(t) = t^n & \rightarrow A'(t) = n t^{n-1} \\ B'(t) = e^{-t} & \rightarrow B(t) = -e^{-t} \end{cases}$$

$$\Rightarrow G_n(u) = \left[-t^n e^{-t} \right]_0^u + n \int_0^u t^{n-1} e^{-t} dt$$

$$\Rightarrow \boxed{G_n(u) = -u^n e^{-u} + n \cdot G_{n-1}(u)}$$

$$b) \text{ Ma } \forall n \geq 2, \quad G_n(u) = -n! e^{-u} \sum_{k=0}^n \frac{u^k}{k!} + n! G_1(u)$$

$$\text{Pour } n=2, \quad G_2(u) = -2! e^{-u} \frac{u^2}{2!} + 2! G_1(u)$$

$$\text{ma } G_2(u) = -u^2 e^{-u} + 2 G_1(u) \Rightarrow \text{vérifier}$$

Suppose $G_n(u) = -n! \cdot e^{-u} \sum_{p=2}^n \frac{u^p}{p!} + n! \cdot G_1(u)$

Def $G_{n+1}(u) = -(n+1)! \cdot e^{-u} \sum_{p=2}^{n+1} \frac{u^p}{p!} + (n+1)! \cdot G_1(u)$

ex eff: $G_{n+1}(u) = -u^{n+1} \cdot e^{-u} + (n+1) \cdot G_n(u)$

$$\begin{aligned} G_{n+1}(u) &= -u^{n+1} \cdot e^{-u} + (n+1) \left[-n! \cdot e^{-u} \sum_{p=2}^n \frac{u^p}{p!} + n! \cdot G_1(u) \right] \\ &= -u^{n+1} \cdot e^{-u} - (n+1)! \cdot e^{-u} \sum_{p=2}^n \frac{u^p}{p!} + (n+1)! \cdot G_1(u) \\ &= -(n+1)! \cdot e^{-u} \left[\sum_{p=2}^n \frac{u^p}{p!} + \frac{u^{n+1}}{(n+1)!} \right] + (n+1)! \cdot G_1(u) \end{aligned}$$

$$\Rightarrow G_{n+1}(u) = -(n+1)! \cdot e^{-u} \sum_{p=2}^{n+1} \frac{u^p}{p!} + (n+1)! \cdot G_1(u)$$

Cond: $\forall n \geq 2 \quad \forall u \geq 0$

$$G_n(u) = -n! \cdot e^{-u} \sum_{p=2}^n \frac{u^p}{p!} + n! \cdot G_1(u)$$

2) Def $\lim_{u \rightarrow +\infty} G_n(u) = n!$

ona $G_n(u) = -n! \cdot e^{-u} \sum_{p=2}^n \frac{u^p}{p!} + n! \cdot G_1(u)$

$$\Rightarrow G_n(u) = -n! \cdot \sum_{p=2}^n \frac{u^{p-1}}{p!} + n! \cdot G_1(u)$$

or $\lim_{u \rightarrow +\infty} u^{p-1} = \lim_{u \rightarrow +\infty} \frac{u^p}{e^u} = \lim_{u \rightarrow +\infty} \frac{1}{\frac{e^u}{u^p}} = 0$

$$\Rightarrow \lim_{u \rightarrow +\infty} \frac{u^{p-1}}{p!} = \frac{0}{p!} = 0 \quad \forall p \in \mathbb{N}^+$$

$$\Rightarrow \lim_{u \rightarrow +\infty} \sum_{p=2}^n \frac{u^{p-1}}{p!} = \sum_{p=2}^n \lim_{u \rightarrow +\infty} \left(\frac{u^{p-1}}{p!} \right)$$

$$= \sum_{p=2}^n 0 = (n-1) \cdot 0 = 0$$

car $n-1$: constante.

$$\Rightarrow \lim_{u \rightarrow +\infty} -n! \cdot \sum_{p=2}^n \frac{u^{p-1}}{p!} = (-n!) \cdot 0 = 0$$

Def $G_1(u) = \int_0^u t e^{-t} dt = \int_0^u f_1(t) dt = S_u$

$$\Rightarrow \lim_{u \rightarrow +\infty} G_1(u) = \lim_{u \rightarrow +\infty} S_u = 1 \quad (\text{d'après A})$$

et $\lim_{u \rightarrow +\infty} (n!) = n! : \text{constante}$

$$\Rightarrow \lim_{u \rightarrow +\infty} G_n(u) = n!$$

3) a) On $\forall n \in \mathbb{N}^+ \quad G'_n(nx) = n^n \cdot f_n(x)$

ona $G_n(nx) = \int_0^{nx} t^n e^{-t} dt$

or $h: t \mapsto t^n e^{-t} \in \mathcal{BM}(\mathbb{R}_+) \Rightarrow$ admet une primitive $H \in \mathcal{BM}(\mathbb{R}_+) / H'(t) = h(t)$

d'ou $G_n(nx) = [H(t)]_0^{nx} = H(nx) - H(0)$

Comme $nx \in \mathcal{BM}(\mathbb{R}_+) \Rightarrow G_n \in \mathcal{BM}(\mathbb{R}_+)$

$$G'_n(nx) \in \mathcal{BM}(\mathbb{R}_+) = \mathcal{D}_H$$

$$[G_n(nx)]' = [H(nx)]' - [H(0)]'$$

$$\Rightarrow G'_n(nx) = n \cdot H'(nx) - 0 = n \cdot h(nx)$$

$$\Rightarrow G'_n(nx) = (nx)^n e^{-nx} = n^n \cdot x^n e^{-nx}$$

et $\forall x \quad G'_n(nx) = n^n \cdot f_n(x)$

b) Def $F_n(x) = \frac{1}{n^{n+1}} G_n(nx)$

ona $G'_n(nt) = n^n f_n(t) \Rightarrow f_n(t) = \frac{1}{n^n} G'_n(nt)$

$$\Rightarrow \int_0^x f_n(t) dt = \frac{1}{n^n} \int_0^x G'_n(nt) dt$$

$$\Rightarrow F_n(x) = \frac{1}{n^n} \left[\frac{1}{n} G_n(nt) \right]_0^x$$

$$\Rightarrow F_n(x) = \frac{1}{n^{n+1}} [G_n(nx) - G_n(0)]$$

or $G_n(0) = \int_0^0 h(t) dt = 0$

$$\Rightarrow \forall x \geq 0, \quad F_n(x) = \frac{1}{n^{n+1}} G_n(x)$$

c) $\lim_{x \rightarrow +\infty} G_n(x) = \lim_{u \rightarrow +\infty} G_n(u) = n!$

$$\Rightarrow \lim_{x \rightarrow +\infty} F_n(x) = \frac{n!}{n^{n+1}}$$

Problème 14.: BAC 1998 - Session Principale

A) $f(x) = \ln(2x + \sqrt{4x^2 + 1})$

1) a) f existe ssi $\begin{cases} 4x^2 + 1 \geq 0 \\ 2x + \sqrt{4x^2 + 1} > 0 \end{cases}$

ou $1 > 0 \Rightarrow 4x^2 + 1 > 4x^2$

$\Rightarrow \sqrt{4x^2 + 1} > |2x| \Leftrightarrow \begin{cases} \sqrt{4x^2 + 1} > 2x \\ \text{et} \\ \sqrt{4x^2 + 1} > -2x \end{cases}$

$\Rightarrow \begin{cases} \sqrt{4x^2 + 1} - 2x > 0 \\ \text{et} \\ \sqrt{4x^2 + 1} + 2x > 0 \end{cases} \quad \forall x \in \mathbb{R}$

$\Rightarrow \boxed{Df = \mathbb{R}}$ d'où $(-x) \in \mathbb{R}$

$f(x) + f(-x) = \ln(\sqrt{4x^2 + 1} + 2x) + \ln(\sqrt{4x^2 + 1} - 2x)$

$= \ln[(\sqrt{4x^2 + 1} + 2x)(\sqrt{4x^2 + 1} - 2x)]$

$= \ln(4x^2 + 1 - 4x^2) = \ln 1 = 0$

$\Rightarrow f(-x) = -f(x) \Rightarrow f$: impaire

b) car $\forall x \in \mathbb{R}, \begin{cases} 4x^2 + 1 > 0 \\ 2x + \sqrt{4x^2 + 1} > 0 \end{cases}$

$\Rightarrow f$ est dérivable sur $\mathbb{R}$

$f'(x) = \frac{(2x + \sqrt{4x^2 + 1})'}{2x + \sqrt{4x^2 + 1}} = \frac{2 + \frac{8x}{2\sqrt{4x^2 + 1}}}{2x + \sqrt{4x^2 + 1}}$

$= \frac{2\sqrt{4x^2 + 1} + 4x}{(2x + \sqrt{4x^2 + 1})\sqrt{4x^2 + 1}} = \frac{2(\sqrt{4x^2 + 1} + 2x)}{(2x + \sqrt{4x^2 + 1})\sqrt{4x^2 + 1}}$

$\Rightarrow \forall x \in \mathbb{R}, \boxed{f'(x) = \frac{2}{\sqrt{4x^2 + 1}}}$

2) a) $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \ln(2x + \sqrt{4x^2 + 1}) = +\infty$

$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\ln(2x + \sqrt{4x^2 + 1})}{x}$

$= \lim_{x \rightarrow +\infty} \ln\left(x \left(2 + \sqrt{4 + \frac{1}{x^2}}\right)\right)$

$= \lim_{x \rightarrow +\infty} \left(\ln x + \ln\left(2 + \sqrt{4 + \frac{1}{x^2}}\right) \right) = 0$

$\Rightarrow E_f$ admet une branche infinie au voisinage de $(+\infty)$ de direction $(\vec{ot})$

Comme f est impaire $\Rightarrow$ par symétrie par rapport à O , E_f admet une autre branche infinie au voisinage de $(-\infty)$ de direction $(\vec{ot})$ aussi

b) $T_0: y = f'(0)(x - 0) + f(0)$

ou $f'(0) = 2$ et $f(0) = 0$

$\Rightarrow T_0: \boxed{y = 2x}$

c) Position de E_f et T_0

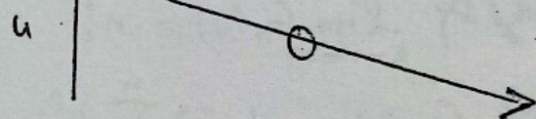
on pose $u(x) = f(x) - y = f(x) - 2x$

$D_u = \mathbb{R}, u'(x) = f'(x) - 2 = \frac{2}{\sqrt{4x^2 + 1}} - 2$

$\Rightarrow u'(x) = \frac{2 - 2\sqrt{4x^2 + 1}}{\sqrt{4x^2 + 1}} = \frac{-8x^2}{\sqrt{4x^2 + 1}(2 + 2\sqrt{4x^2 + 1})}$

$\Rightarrow \forall x \in \mathbb{R}, u'(x) \leq 0$

x	$-\infty$	0	$+\infty$
u'	-	0	-



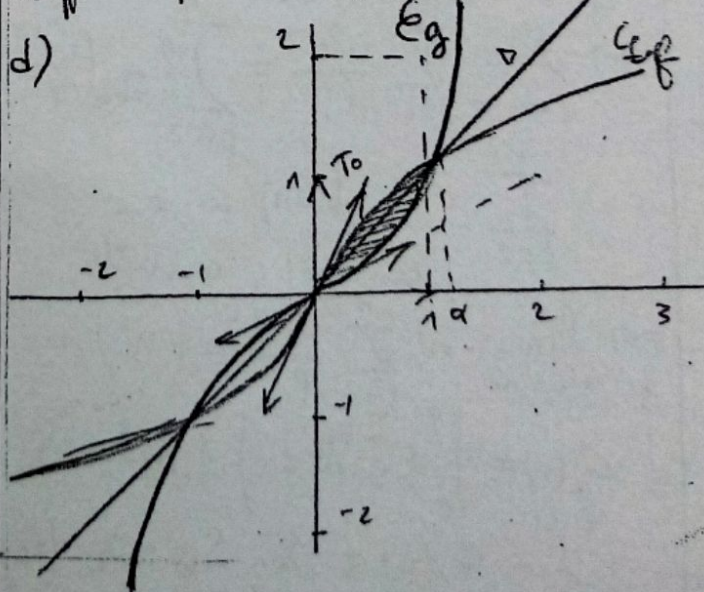
$u(0) = 0 \Rightarrow \forall x \in]-\infty, 0[\Rightarrow u(x) \geq 0$

et $\forall x \geq 0, u(x) \leq 0$

x	$-\infty$	0	$+\infty$
u(x)	+	0	-

Position de E_f par rapport à T_0

E_f est au dessus de T_0 $E_f \cap T_0$ E_f est au dessous de T_0



$$3) \lambda > 0 \quad A(\lambda) = \int_0^\lambda f(x) dx = \int_0^\lambda \ln(2x + \sqrt{4x^2 + 1}) dx \quad \lim_{x \rightarrow +\infty} V(x) = \lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{x \rightarrow +\infty} x \left(\frac{f(x)}{x} - 1 \right)$$

$$\text{on pose } u(x) = \ln(2x + \sqrt{4x^2 + 1}) \rightarrow u'(x) = \frac{2}{\sqrt{4x^2 + 1}}$$

$$v'(x) = 1 \rightarrow v(x) = x$$

$$\Rightarrow A(\lambda) = \left[x f(x) \right]_0^\lambda - \int_0^\lambda \frac{2x}{\sqrt{4x^2 + 1}} dx$$

$$\Rightarrow A(\lambda) = \left[x f(x) \right]_0^\lambda - \frac{1}{2} \int_0^\lambda \frac{8x}{2\sqrt{4x^2 + 1}} dx$$

$$\Rightarrow A(\lambda) = \left[x f(x) - \frac{1}{2} \sqrt{4x^2 + 1} \right]_0^\lambda$$

$$\Rightarrow A(\lambda) = \left(\lambda f(\lambda) - \frac{1}{2} \sqrt{4\lambda^2 + 1} \right) - \left(0 - \frac{1}{2} \right)$$

$$\Rightarrow A(\lambda) = \frac{1}{2} + \lambda f(\lambda) - \frac{1}{2} \sqrt{4\lambda^2 + 1} \quad (u.a.)$$

5) 1) a) $f \in \text{strict } \nearrow \text{ sur } \mathbb{R} \Rightarrow f \text{ réalise une bijection de } \mathbb{R} \text{ sur } \mathbb{R} = I$

$$b) E_{f^{-1}} = E_g = S_D(E_f) \quad / \Delta: y = x$$

$$c) g = f^{-1} \text{ définie par: } \begin{cases} \forall y \in \mathbb{R} \\ f(y) = x \end{cases} \Leftrightarrow \begin{cases} \forall x \in \mathbb{R} \\ y = g(x) \end{cases}$$

$$\text{en effet: } f(y) = x \Leftrightarrow \ln(2y + \sqrt{4y^2 + 1}) = x$$

$$\Leftrightarrow 2y + \sqrt{4y^2 + 1} = e^x \Leftrightarrow \sqrt{4y^2 + 1} = e^x - 2y$$

$$\Leftrightarrow 4y^2 + 1 = (e^x - 2y)^2 = e^{2x} - 4ye^x + 4y^2$$

$$\Leftrightarrow 4ye^x = e^{2x} - 1 \Leftrightarrow y = \frac{e^{2x} - 1}{4e^x}$$

$$\Rightarrow g(x) = \frac{1}{4} (e^x - e^{-x}) \quad \forall x \in \mathbb{R}$$

$$2) a) g(x) = x \Leftrightarrow f(x) = x \Leftrightarrow f(x) - x = 0$$

$$\text{on pose } V(x) = f(x) - x \quad D_V = \mathbb{R}_+^*$$

$$V \in \text{sur }]0, +\infty[, \quad V'(x) = f'(x) - 1$$

$$\Leftrightarrow V'(x) = \frac{2}{\sqrt{4x^2 + 1}} - 1 = \frac{2 - \sqrt{4x^2 + 1}}{\sqrt{4x^2 + 1}}$$

$$\Leftrightarrow V'(x) = \frac{4 - (4x^2 + 1)}{\sqrt{4x^2 + 1} (2 + \sqrt{4x^2 + 1})} = \frac{3 - 4x^2}{D} > 0$$

$$V'(x) = 0 \Leftrightarrow 4x^2 = 3 \Leftrightarrow \begin{cases} x^2 = \frac{3}{4} \\ x > 0 \end{cases} \Rightarrow x = \frac{\sqrt{3}}{2}$$

$$\lim_{x \rightarrow 0^+} V(x) = \lim_{x \rightarrow 0^+} (f(x) - x) = 0$$

$$V\left(\frac{\sqrt{3}}{2}\right) = f\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} = \ln(\sqrt{3} + 2) - \frac{\sqrt{3}}{2} > 0$$

x	0		$\frac{\sqrt{3}}{2}$	α	$+\infty$
V'(x)		+	0	-	
V(x)	0		$V\left(\frac{\sqrt{3}}{2}\right)$		$-\infty$

• si $x \in]0, \frac{\sqrt{3}}{2}[$, $V \in \text{strict } \nearrow$
 et $0 \notin]0, V(\frac{\sqrt{3}}{2})] \Rightarrow$ l'éq $V(x) = 0$ n'admet pas de solution.

• si $x \in [\frac{\sqrt{3}}{2}, +\infty[$, $V \in \text{strict } \searrow$
 et $V\left(\frac{\sqrt{3}}{2}, +\infty\right[=]-\infty, V(\frac{\sqrt{3}}{2})]$
 or $0 \in V(\frac{\sqrt{3}}{2}) = \exists ! \alpha \geq \frac{\sqrt{3}}{2} / V(\alpha) = 0$

$$\Leftrightarrow f(\alpha) - \alpha = 0 \Leftrightarrow f(\alpha) = \alpha \Leftrightarrow g(\alpha) = \alpha$$

$$b) A = \int_0^\alpha (f(x) - g(x)) dx = 2 \int_0^\alpha (f(x) - x) dx$$

$$= 2 \int_0^\alpha f(x) dx - \int_0^\alpha 2x dx$$

$$= 2A(\alpha) - [x^2]_0^\alpha$$

$$A = 1 + 2\alpha f(\alpha) - \sqrt{4\alpha^2 + 1} - \alpha^2$$

$$\text{or } f(\alpha) = \alpha \Rightarrow A = (\alpha^2 + 1 - \sqrt{4\alpha^2 + 1}) \quad (u.a.)$$

$$c) \varphi(x) = \frac{g(x)}{g'(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

$$1) a) \varphi \in \text{strict } \nearrow \text{ sur } \mathbb{R} \text{ car } e^{2x} + 1 \neq 0$$

$$\varphi'(x) = \frac{2e^{2x}(e^{2x} + 1) - e^{2x}(e^{2x} - 1)}{(e^{2x} + 1)^2} = \frac{4e^{2x}}{(e^{2x} + 1)^2}$$

$$\lim_{x \rightarrow -\infty} \varphi(x) = \lim_{x \rightarrow -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = -1 \text{ car } \lim_{x \rightarrow -\infty} e^{2x} = 0$$

$$\lim_{x \rightarrow +\infty} \varphi(x) = \lim_{x \rightarrow +\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1 \text{ car } \lim_{x \rightarrow +\infty} e^{-2x} = 0$$

x	$-\infty$	$+\infty$
φ		$\varphi \in \text{strict } \nearrow \text{ sur } \mathbb{R}$
φ'	+	$\Rightarrow \varphi$ réalise une bijection de $\mathbb{R}$
		sur $I =]-1, 1[$

b) φ réalise une bijection $\Rightarrow$
 φ admet une fonction réciproque $\varphi^{-1} = h$
 définie par : $\begin{cases} \forall y \in \mathbb{R} \\ \varphi(y) = x \end{cases} \Leftrightarrow \begin{cases} \forall x \in]-1,1[\\ y = h(x) \end{cases}$

en effet : $\varphi(y) = x \Leftrightarrow \frac{e^{2y}-1}{e^{2y}+1} = x$

$\Leftrightarrow e^{2y}-1 = xe^{2y}+x \Leftrightarrow e^{2y}(1-x) = 1+x$

$\Leftrightarrow e^{2y} = \frac{1+x}{1-x} > 0, \forall x \in]-1,1[$

$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

on $y = h(x) \Rightarrow \boxed{h(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)}$

$h'(x) = \frac{1}{2} \cdot \frac{\frac{2}{1-x} \cdot \frac{1-x}{1+x}}{\frac{1-x}{1+x}} = \frac{1}{2} \cdot \frac{2}{(1-x)^2} \cdot \frac{1-x}{1+x}$

$\Rightarrow h'(x) = \frac{1}{(1-x)(1+x)} \Rightarrow \boxed{h'(x) = \frac{1}{1-x^2}}$

2) $\forall x \in]0,1[$ et $n \in \mathbb{N}^*$

$S_n(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{x^{2n-1}}{2n-1}$

a) on a $S_n(x)$ p.s. $\equiv$ sur $]0,1[$

$S'_n(t) = 1 + t^2 + t^4 + t^6 + \dots + t^{2n-2}$
 $= 1 + t^2 + (t^2)^2 + (t^2)^3 + \dots + (t^2)^{n-1}$

$\Rightarrow S'_n(t) = \sum_{k=0}^{n-1} (t^2)^k = \frac{1-(t^2)^n}{1-t^2} = \frac{1-t^{2n}}{1-t^2}$

$\Rightarrow S'_n(t) = \frac{1}{1-t^2} - \frac{t^{2n}}{1-t^2} = h'(t) - \frac{t^{2n}}{1-t^2}$

on $S'_n(t) \in m[0, \pi]$ / $x \in]0,1[$

et $h'(t) \in m[0, \pi]$

$\frac{t^{2n}}{1-t^2} \in m[0, \pi]$

$\int_0^x S'_n(t) dt = \int_0^x h'(t) dt - \int_0^x \frac{t^{2n}}{1-t^2} dt$

$\Rightarrow [S_n(t)]_0^x = [h(t)]_0^x - \int_0^x \frac{t^{2n}}{1-t^2} dt$

$\Rightarrow S_n(x) - S_n(0) = h(x) - h(0) - \int_0^x \frac{t^{2n}}{1-t^2} dt$

on $S_n(0) = 0$ et $h(0) = 0$

$\Rightarrow \forall x \in]0,1[, S_n(x) = h(x) - \int_0^x \frac{t^{2n}}{1-t^2} dt$

b) $U_n = \frac{1}{3} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \dots + \frac{1}{(2n-1)3^{2n-1}}$

$\Rightarrow U_n = \frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \dots + \frac{(\frac{1}{3})^{2n-1}}{2n-1}$

$\Rightarrow U_n = S_n\left(\frac{1}{3}\right) = h\left(\frac{1}{3}\right) - \int_0^{\frac{1}{3}} \frac{t^{2n}}{1-t^2} dt$

on $0 \leq t \leq \frac{1}{3} \Rightarrow 0 \leq t^2 \leq \frac{1}{9}$

$\Rightarrow -\frac{1}{9} \leq -t^2 \leq 0 \Rightarrow \frac{8}{9} \leq 1-t^2 \leq 1$

$\Rightarrow 1 \leq \frac{1}{1-t^2} \leq \frac{9}{8} \Rightarrow 0 \leq \frac{1}{1-t^2} \leq \frac{9}{8}$

$t^{2n} \geq 0 \Rightarrow 0 \leq \frac{t^{2n}}{1-t^2} \leq \frac{9}{8} t^{2n}$

Comme $0 < \frac{1}{3} \Rightarrow 0 \leq \int_0^{\frac{1}{3}} \frac{t^{2n}}{1-t^2} dt \leq \frac{9}{8} \int_0^{\frac{1}{3}} t^{2n} dt$

$\Rightarrow 0 \leq \int_0^{\frac{1}{3}} \frac{t^{2n}}{1-t^2} dt \leq \frac{9}{8} \left[\frac{t^{2n+1}}{2n+1} \right]_0^{\frac{1}{3}}$

$\Rightarrow 0 \leq \int_0^{\frac{1}{3}} \frac{t^{2n}}{1-t^2} dt \leq \frac{9}{8} \cdot \frac{(\frac{1}{3})^{2n+1}}{2n+1}$

$\Rightarrow -\frac{9}{8} \frac{(\frac{1}{3})^{2n+1}}{2n+1} \leq -\int_0^{\frac{1}{3}} \frac{t^{2n}}{1-t^2} dt \leq 0$

on $h\left(\frac{1}{3}\right) = \frac{1}{2} \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{1}{2} \ln\left(\frac{4}{2}\right) = \frac{1}{2} \ln 2$

$\Rightarrow \frac{1}{2} \ln 2 - \frac{9}{8} \frac{(\frac{1}{3})^{2n+1}}{2n+1} \leq h\left(\frac{1}{3}\right) - \int_0^{\frac{1}{3}} \frac{t^{2n}}{1-t^2} dt \leq \frac{1}{2} \ln 2$

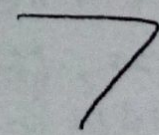
$\Rightarrow \frac{1}{2} \ln 2 - \frac{9}{8} \frac{(\frac{1}{3})^{2n+1}}{2n+1} \leq U_n \leq \frac{1}{2} \ln 2$

on $\frac{1}{3} \in]-1,1[\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{3}\right)^{2n+1} = 0$

Ainsi $\lim_{n \rightarrow \infty} \frac{9}{8} \frac{(\frac{1}{3})^{2n+1}}{2n+1} = \frac{0}{\infty} = 0$

$\Rightarrow \lim U_n = \frac{1}{2} \ln 2$

concl : $\boxed{U_n \text{ converge vers } \ln(\sqrt{2})}$



Exercice 15 : BAC 1998 - Session de Contrôle

$$f(x) = \sqrt{x^2 - 4x + 5} \quad D_f = \mathbb{R}$$

1) a) $f \in \mathcal{C}^1$ sur $\mathbb{R}$ car $x^2 - 4x + 5 > 0$

$$f'(x) = \frac{2x - 4}{2\sqrt{x^2 - 4x + 5}} = \frac{x - 2}{\sqrt{x^2 - 4x + 5}}$$

x	$-\infty$	2	$+\infty$
f'(x)	-	0	+
f(x)	$+\infty$	1	$+\infty$

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \sqrt{x^2 - 4x + 5} = +\infty \quad (B.I.)$$

$$f(2) = \sqrt{4 - 8 + 5} = 1$$

$$b) \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1 - \frac{4}{x} + \frac{5}{x^2}}}{1} = 1$$

$$\lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{x \rightarrow +\infty} (\sqrt{x^2 - 4x + 5} - x)$$

$$= \lim_{x \rightarrow +\infty} \frac{x^2 - 4x + 5 - x^2}{\sqrt{x^2 - 4x + 5} + x} = \lim_{x \rightarrow +\infty} \frac{-4x + 5}{x(\sqrt{1 - \frac{4}{x} + \frac{5}{x^2}} + 1)} = -2$$

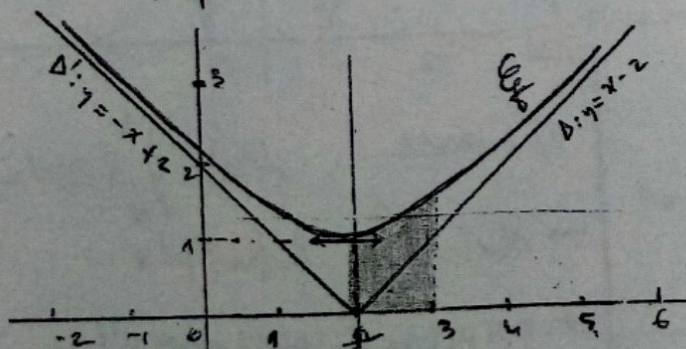
$\Rightarrow D: y = x - 2$ A.O en $(+\infty)$.

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} \frac{(1x)\sqrt{1 - \frac{4}{x} + \frac{5}{x^2}}}{x} = -1$$

$$\lim_{x \rightarrow -\infty} (f(x) + x) = \lim_{x \rightarrow -\infty} (\sqrt{x^2 - 4x + 5} + x)$$

$$= \lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 5 - x^2}{-x\sqrt{1 - \frac{4}{x} + \frac{5}{x^2}} + x} = \lim_{x \rightarrow -\infty} \frac{-4x + 5}{x(-\sqrt{1 - \frac{4}{x} + \frac{5}{x^2}} + 1)} = 2$$

$\Rightarrow D: y = -x + 2$ A.O en $(-\infty)$



$$2) \mathcal{A}(2,0), M(x,y) \in \mathcal{C} \Rightarrow \vec{OM} = x\vec{u} + y\vec{v}$$

$$M(x,y) \in \mathcal{C}' \Rightarrow \vec{OM} = x\vec{u} + y\vec{v}$$

$$\text{or } \vec{OM} = \vec{OA} + \vec{AM} = 2\vec{u} + x\vec{u} + y\vec{v}$$

Par identifi: $\begin{cases} x = x + z \\ y = y \end{cases}$

$$\text{or } M(x,y) \in \mathcal{C}_f \Rightarrow y = f(x) = \sqrt{x^2 - 4x + 5}$$

$$\Rightarrow y = \sqrt{(x+2)^2 - 4(x+2) + 5} = \sqrt{x^2 + 4x + 4 - 4x - 8 + 5}$$

$$\Rightarrow \boxed{y = \sqrt{x^2 + 1}} \text{ eq. de } \mathcal{C} \text{ dans } \mathcal{R}'$$

$$b) M(x,y) \in \mathcal{C}_g \Rightarrow y = g(x)$$

$$\Rightarrow y = \sqrt{x^2 + 1} \Rightarrow \mathcal{C}_g = \mathcal{C}_f \text{ dans } \mathcal{R}'$$

$$3) F(x) = \int_0^x \frac{1}{g(t)} dt \quad D_F = \mathbb{R}$$

$$a) \frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}}) = 1 \Rightarrow (e^{\frac{x}{2}} - e^{-\frac{x}{2}} = 2) e^{\frac{x}{2}}$$

$$\Rightarrow e^x - 2e^{\frac{x}{2}} - 1 = 0$$

$$D' = 1 + 1 = 2 \Rightarrow e^{\frac{x}{2}} = 1 + \sqrt{2} < 0 \text{ Absolu}$$

$$e^{-\frac{x}{2}} = 1 + \sqrt{2} \Rightarrow \frac{x}{2} = \ln(1 + \sqrt{2}) \Rightarrow x = 2\ln(1 + \sqrt{2})$$

$$\Rightarrow S_{\mathbb{R}} = \{2\ln(1 + \sqrt{2})\}$$

$$b) \text{ on a } x \mapsto \frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}}) \in \text{an } \mathbb{R}$$

$$\text{et } g \in \text{an } \mathbb{R} \Rightarrow F \in \text{an } \mathbb{R}$$

$$F'(x) = \left[\frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}}) \right]' \cdot g\left(\frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}})\right)$$

$$\Rightarrow F'(x) = \frac{1}{4} (e^{\frac{x}{2}} + e^{-\frac{x}{2}}) \cdot \sqrt{\frac{1}{4} (e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2 + 1}$$

$$\Rightarrow F'(x) = \frac{1}{8} (e^{\frac{x}{2}} + e^{-\frac{x}{2}}) \cdot \sqrt{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2 + 4}$$

$$= \frac{1}{8} (e^{\frac{x}{2}} + e^{-\frac{x}{2}}) \sqrt{(e^{\frac{x}{2}} + e^{-\frac{x}{2}})^2}$$

$$\Rightarrow F'(x) = \frac{1}{8} (e^{\frac{x}{2}} + e^{-\frac{x}{2}})^2 = \frac{1}{8} (e^x + e^{-x} + 2)$$

$$c) A = \int_2^3 f(t) dt = \int_0^1 g(x) dx$$

$$\text{con } (x=2)_{\mathcal{R}} \Rightarrow (x=0)_{\mathcal{R}'}$$

$$\text{et } (x=3)_{\mathcal{R}} \Rightarrow (x=1)_{\mathcal{R}'}$$

$$\Rightarrow A = F(2\ln(1 + \sqrt{2})) = F(\ln((1 + \sqrt{2})^2))$$

$$\text{or } F(x) = \frac{1}{8} (e^x - e^{-x} + 2x) + k$$

$$\text{con } F(0) = 0 \Rightarrow k = 0$$

$$\Rightarrow A = \frac{1}{8} (e^{\ln((1 + \sqrt{2})^2)} - e^{-\ln((1 + \sqrt{2})^2)} + 2\ln((1 + \sqrt{2})^2))$$

$$= \frac{1}{8} ((1 + \sqrt{2})^2 - \frac{1}{(1 + \sqrt{2})^2} + 4\ln(1 + \sqrt{2}))$$

$$\Rightarrow A = \frac{(1 + \sqrt{2})^2}{8} - \frac{1}{8(1 + \sqrt{2})^2} + \frac{\ln(1 + \sqrt{2})}{2} \quad (\text{ua})$$

Problème 16: BAC 1997 - Session Principale

$$D =]-\infty, -1[\cup]0, +\infty[\quad \text{et} \quad D^* =]-\infty, -1[\cup]0, +\infty[$$

Ⓐ $f(x) = x \cdot \ln\left(1 + \frac{1}{x}\right)$ si $x \in D^*$ et $f(0) = 0$

1a) $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \cdot \ln\left(1 + \frac{1}{x}\right) = \lim_{x \rightarrow 0^+} x \cdot \ln\left(\frac{x+1}{x}\right)$
 $= \lim_{x \rightarrow 0^+} x [\ln(x+1) - \ln x] = \lim_{x \rightarrow 0^+} [x \ln(x+1) - x \ln x] = 0$
 $= f(0) \Rightarrow f$ est continue en 0^+

b) $\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x \cdot \ln\left(1 + \frac{1}{x}\right)}{x} = +\infty$

$\Rightarrow f$ n'est pas $\leq$ en 0^+ et $\exists$ une tangente verticale dirigée vers le haut.
 une demi-tangente verticale dirigée vers le haut.

2a) $f \leq$ si $x \neq 0$ et $1 + \frac{1}{x} > 0 \Leftrightarrow \frac{x+1}{x} > 0$

x	$-\infty$	-1	0	$+\infty$
$\frac{x+1}{x}$	$-$	0^+	$+$	$+$
x	$-$	$-$	0^+	$+$
$\frac{x+1}{x^2}$	$+$	0^-	$+$	$+$

$\Rightarrow f \leq$ sur $]-\infty, -1[\cup]0, +\infty[$
 d'où $f \leq$ sur D^*

$f'(x) = \left[x \ln\left(1 + \frac{1}{x}\right) \right]' = \ln\left(1 + \frac{1}{x}\right) + x \cdot \frac{-\frac{1}{x^2}}{1 + \frac{1}{x}}$

$\Leftrightarrow f'(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}$

b) variations de f' : $f''(x) = \frac{-\frac{1}{x^2}}{1 + \frac{1}{x}} + \frac{1}{(x+1)^2}$

$\Leftrightarrow f''(x) = \frac{-1}{x(x+1)} + \frac{1}{(x+1)^2} = \frac{-1}{x(x+1)^2}$

$\text{sgn}(f'') = \text{sgn}\left[-\frac{1}{x}\right] = \text{sgn}[-x]$

x	$-\infty$	-1	0	$+\infty$
$f''(x)$	$+$	0^-	$-$	$-$
$f'(x)$	$+$	0^+	0^-	$+$

$\lim_{x \rightarrow -\infty} f'(x) = 0$
 $\lim_{x \rightarrow 0^+} f'(x) = +\infty$

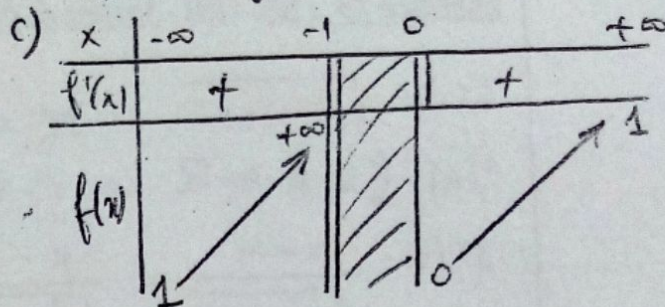
$\lim_{x \rightarrow (-1)^-} f'(x) = \lim_{(-1)^-} \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1} \right] = \lim_{(-1)^-} [\ln(-1-x) - \ln(-x)] + \frac{1}{-1-1}$
 $= \lim_{(-1)^-} \frac{(-x-1)\ln(-x-1) + 1}{(-x-1)} - \ln(-x)$

on pose $t = -x-1$ si $x \rightarrow (-1)^- \Rightarrow t \rightarrow 0^+$

$\lim_{t \rightarrow 0^+} \frac{t \ln t + 1}{t} - \ln(t+1) = +\infty$

d'après le TV de f' on a:

$\forall x \in D^* \quad f'(x) > 0$



$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{t \rightarrow 0} x \ln\left(1 + \frac{1}{x}\right)$

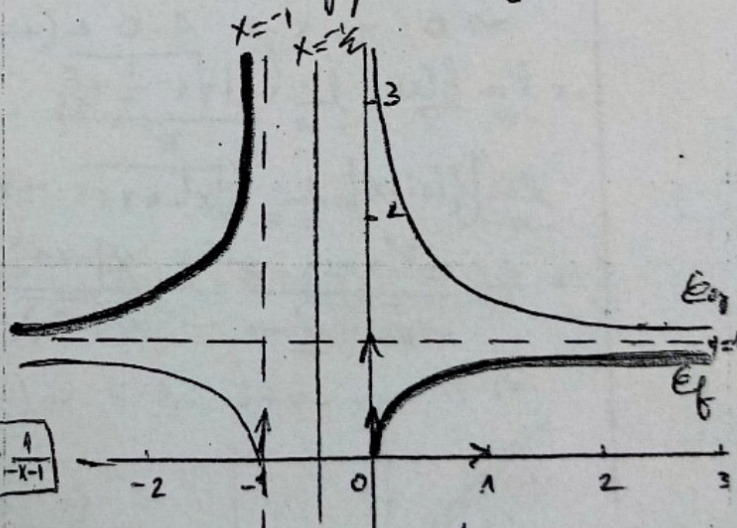
on pose $t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

si $x \rightarrow \pm\infty \Rightarrow t \rightarrow 0$

$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{t \rightarrow 0} \frac{\ln(1+t)}{t} = 1$

$\lim_{x \rightarrow (-1)^+} f(x) = \lim_{(-1)^+} x \ln\left(1 + \frac{1}{x}\right)$
 $= (-1) \cdot (-\infty) = +\infty$

d'où $D_1: x = -1$ Asymptote verticale
 et $D_2: y = 1$: asymptote horiz en $(\pm\infty)$



3) $0 < \alpha < 1$, $A(\alpha) = \int_{\alpha}^1 f(x) dx$

$\Leftrightarrow A(\alpha) = \int_{\alpha}^1 x \cdot \ln\left(1 + \frac{1}{x}\right) dx$

on pose $u(x) = \ln\left(1 + \frac{1}{x}\right) \rightarrow u'(x) = \frac{-1}{x(x+1)}$
 $v(x) = x \rightarrow v'(x) = 1$

$\Rightarrow A(\alpha) = \left[\frac{1}{2} x^2 \ln\left(1 + \frac{1}{x}\right) \right]_{\alpha}^1 + \frac{1}{2} \int_{\alpha}^1 \frac{x+1-1}{x+1} dx$

$$\Rightarrow A(x) = \left[\frac{1}{2} x^2 \ln \left(1 + \frac{1}{x} \right) \right]_{\alpha}^1 + \frac{1}{2} \left[x - \ln(x+1) \right]_{\alpha}^1$$

$$\Rightarrow A(x) = \frac{1}{2} \ln 2 - \frac{1}{2} \alpha f(x) + \frac{1}{2} \ln 2 - \frac{1}{2} \alpha + \frac{1}{2} \ln(\alpha+1)$$

$$\Rightarrow A(x) = \frac{1}{2} - \frac{1}{2} \alpha - \frac{1}{2} \alpha f(x) + \frac{1}{2} \ln(\alpha+1) \quad (ua)$$

$$b) \lim_{\alpha \rightarrow 0} A(x) = \frac{1}{2} \quad (ua) \text{ car } f(0) = 0$$

B $\Delta: x = -\frac{1}{2}$

1) $E' = S_{\Delta}(E) /$

2) $M(x, y)$ et $M'(x', y')$ $\left. \begin{array}{l} x' = -x-1 \\ y' = y \end{array} \right\}$

a) $\cap_{\Delta} S_{\Delta}(M) = M' \Leftrightarrow$

car $S_{\Delta}(M) = M' \Leftrightarrow \left\{ \begin{array}{l} (MM') \perp \Delta \\ M \neq M' \in \Delta \end{array} \right.$

et $\overrightarrow{MM'} \cdot \begin{pmatrix} x'-x \\ y'-y \end{pmatrix} \cdot \vec{u}_{\Delta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \Rightarrow y' = y$

$$\Rightarrow \boxed{y' = y} \quad (1)$$

et $M \neq M' = \left(\frac{x+x'}{2}, \frac{y+y'}{2} \right) \in \Delta: x = -\frac{1}{2}$

$$\Rightarrow \frac{x+x'}{2} = -\frac{1}{2} \Rightarrow x+x' = -1$$

$$\Rightarrow \boxed{x' = -x-1} \quad (2)$$

(1) et (2) $\rightarrow S_{\Delta}(M) = M' \Leftrightarrow \left\{ \begin{array}{l} x' = -x-1 \\ y' = y \end{array} \right.$

b) $M \in E' \Leftrightarrow M'(x', y') \in S_{\Delta}(E)$

$$\Leftrightarrow M'(x', y') \in E \Leftrightarrow y' = f(x')$$

$$\Rightarrow \boxed{y = f(-x-1)}$$

c) $E' = E_g \Leftrightarrow g(x) = f(-x-1) \Leftrightarrow$

$$g(x) = (-x-1) \ln \left(1 + \frac{1}{-x-1} \right) = (-x-1) \ln \left(\frac{-x}{-x-1} \right)$$

$$\Rightarrow g(x) = (-x-1) \ln \left(\frac{x}{x+1} \right) = -(x+1) \ln \left(\frac{x}{x+1} \right)$$

$$\Rightarrow g(x) = (x+1) \ln \left(\frac{x+1}{x} \right) \Leftrightarrow \boxed{g(x) = (x+1) \ln \left(1 + \frac{1}{x} \right)}$$

C1 $\forall n \in \mathbb{N}^* \quad f(n) < 1 < g(n)$

car $\forall x > 0, f(x) < 1$

et car $\forall n \in \mathbb{N}^* \Rightarrow f(n) < 1$

$\forall x < -1, f(x) \geq 1$

car $x = -t-1 \Rightarrow \forall t > 0, f(-t-1) \geq 1$

d'où $g(t) > 1 \Rightarrow g(n) > 1$

car : $\forall n \in \mathbb{N}^* \quad \boxed{f(n) < 1 < g(n)}$

2) $U_n = \left(1 + \frac{1}{n} \right)^n$ et $V_n = \left(1 + \frac{1}{n} \right)^{n+1}$

a) car $f(n) < 1 < g(n)$

$$\Rightarrow n \cdot \ln \left(1 + \frac{1}{n} \right) < 1 < (n+1) \ln \left(1 + \frac{1}{n} \right)$$

$$\Rightarrow \ln \left[\left(1 + \frac{1}{n} \right)^n \right] < 1 < \ln \left[\left(1 + \frac{1}{n} \right)^{n+1} \right]$$

$$\Rightarrow \left(1 + \frac{1}{n} \right)^n < e < \left(1 + \frac{1}{n} \right)^{n+1}$$

$$\Rightarrow \boxed{U_n < e < V_n}$$

b) $V_n - U_n = \left(1 + \frac{1}{n} \right)^{n+1} - \left(1 + \frac{1}{n} \right)^n = \left(1 + \frac{1}{n} \right)^n \left(\frac{1}{n} \right)$

$$\Rightarrow V_n - U_n = \frac{U_n}{n}$$

et $U_n < e \Rightarrow \frac{U_n}{n} < \frac{e}{n}$ d'où $V_n - U_n < \frac{e}{n}$

c) car $U_n < e < V_n$

$$\Rightarrow 0 < e - U_n < V_n - U_n < \frac{e}{n}$$

d'où $0 < e - U_n < \frac{e}{n}$

et $\lim_{n \rightarrow +\infty} \frac{e}{n} = 0 \Rightarrow \lim_{n \rightarrow +\infty} (e - U_n) = 0$

$$\Rightarrow \boxed{\lim_{n \rightarrow +\infty} U_n = e}$$

et $0 < e - U_n < V_n - U_n < \frac{e}{n}$

d'où $0 < V_n - U_n < \frac{e}{n}$

et $\lim_{n \rightarrow +\infty} \frac{e}{n} = 0 \Rightarrow \lim_{n \rightarrow +\infty} (V_n - U_n) = 0$

$$\Rightarrow \boxed{\lim_{n \rightarrow +\infty} V_n = \lim_{n \rightarrow +\infty} U_n = e}$$

Rq : on peut montrer que $U_n \uparrow$ et $V_n \downarrow$
alors U_n et V_n sont 2 suites
adjacentes et convergent vers la
même limite e

Exercice 17: BAC 1997 - Session de Contrôle

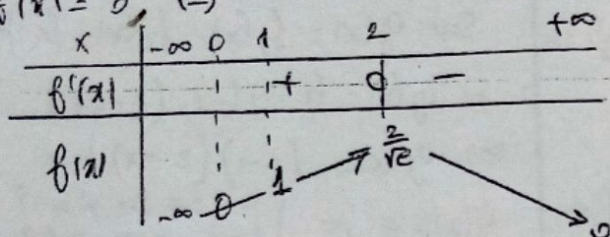
$$f(x) = x \cdot e^{\frac{1-x}{2}} \quad D_f = \mathbb{R}$$

$$1) a) f \in \mathcal{C}^\infty \text{ sur } \mathbb{R}$$

$$f'(x) = e^{\frac{1-x}{2}} + x \cdot \left(-\frac{1}{2}\right) e^{\frac{1-x}{2}}$$

$$\Rightarrow f'(x) = \left(1 - \frac{1}{2}x\right) e^{\frac{1-x}{2}}$$

$$f'(x) = 0 \Rightarrow x = 2$$



$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} x \cdot e^{\frac{1-x}{2}} = -\infty \quad (\text{BSI})$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} x \cdot e^{\frac{1-x}{2}} = \lim_{x \rightarrow +\infty} \frac{x \cdot e^{\frac{1}{2}}}{e^{\frac{x}{2}}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{e}}{e^{\frac{x}{2}}} = \frac{\sqrt{e}}{+\infty} = 0 \Rightarrow \gamma = 0 \text{ A. Hato}$$

$$f(2) = 2e^{-\frac{1}{2}} = \frac{2}{\sqrt{e}} \approx 1,2$$

$$b) \text{ Point d'inflexion: } f' \in \mathcal{C}^\infty \text{ sur } \mathbb{R}$$

$$f''(x) = -\frac{1}{2} e^{\frac{1-x}{2}} - \frac{1}{2} \left(1 - \frac{1}{2}x\right) e^{\frac{1-x}{2}}$$

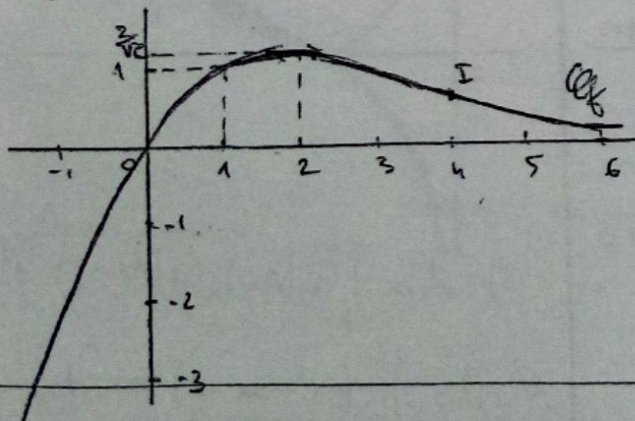
$$\Rightarrow f''(x) = \left(\frac{1}{4}x - 1\right) e^{\frac{1-x}{2}} \quad f'' \mid \begin{array}{c} -\infty \quad 4 \quad +\infty \\ - \quad 0 \quad + \end{array}$$

f'' s'annule en 4 échange de signe.

$$f(4) = 4e^{-\frac{3}{2}} = \frac{4}{e\sqrt{e}} \Rightarrow I\left(4, \frac{4}{e\sqrt{e}}\right) \text{ d'inflexion}$$

$$c) \text{ BI: } \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} e^{\frac{1-x}{2}} = +\infty$$

$\Rightarrow C_f$ est une BI en $-\infty$ de direction $(0, 1)$



$$d) f \text{ sur } (0, 1) \Rightarrow f(0) \leq f(x) \leq f(1)$$

$$\Rightarrow \forall x \in (0, 1), \quad 0 \leq f(x) \leq 1$$

$$2) \forall n \in \mathbb{N}^*, I_n = \frac{1}{n! \cdot 2^{n+1}} \int_0^1 x^n e^{\frac{1-x}{2}} dx$$

$$U_n = 1 + \frac{1}{1! \cdot 2^1} + \frac{1}{2! \cdot 2^2} + \dots + \frac{1}{n! \cdot 2^n}$$

$$a) I_1 = \frac{1}{2^2} \int_0^1 x e^{\frac{1-x}{2}} dx = \int_0^1 \frac{1}{4} x e^{\frac{1-x}{2}} dx$$

$$\text{on pose } \begin{cases} u(x) = \frac{1}{2}x & \rightarrow u'(x) = \frac{1}{2} \\ v'(x) = \frac{1}{2}e^{\frac{1-x}{2}} & \rightarrow v(x) = -e^{\frac{1-x}{2}} \end{cases}$$

$$\Rightarrow I_1 = \left[-\frac{1}{2}x e^{\frac{1-x}{2}} \right]_0^1 + \int_0^1 \frac{1}{2} e^{\frac{1-x}{2}} dx$$

$$\Rightarrow I_1 = \left[\frac{1}{2}x e^{\frac{1-x}{2}} - e^{\frac{1-x}{2}} \right]_0^1 = \left(-\frac{1}{2} - 1 \right) - (0 - e^{\frac{1}{2}}) = \sqrt{e} - \frac{3}{2}$$

$$\cdot I_{n+1} = \frac{1}{(n+1)! \cdot 2^{n+2}} \int_0^1 x^{n+1} e^{\frac{1-x}{2}} dx$$

$$\text{on pose } \begin{cases} u(x) = x^{n+1} & \rightarrow u'(x) = (n+1)x^n \\ v'(x) = e^{\frac{1-x}{2}} & \rightarrow v(x) = -2e^{\frac{1-x}{2}} \end{cases}$$

$$\Rightarrow I_{n+1} = \frac{1}{(n+1)! \cdot 2^{n+2}} \left(\left[-2x^{n+1} e^{\frac{1-x}{2}} \right]_0^1 + 2(n+1) \int_0^1 x^n e^{\frac{1-x}{2}} dx \right)$$

$$\Rightarrow I_{n+1} = \frac{-2}{(n+1)! \cdot 2^{n+2}} + \frac{2(n+1)}{(n+1)! \cdot 2^{n+2}} \int_0^1 x^n e^{\frac{1-x}{2}} dx$$

$$\Rightarrow I_{n+1} = \frac{-1}{(n+1)! \cdot 2^{n+1}} + \frac{1}{n! \cdot 2^{n+1}} \int_0^1 x^n e^{\frac{1-x}{2}} dx$$

$$\Rightarrow I_{n+1} = I_n - \frac{1}{(n+1)! \cdot 2^{n+1}}$$

$$b) \text{ pour } n=1 \quad U_1 = 1 + \frac{1}{1! \cdot 2} = \frac{3}{2} \neq$$

$$I_1 = \sqrt{e} - \frac{3}{2} = \sqrt{e} - U_1 \Rightarrow U_1 = \sqrt{e} - I_1 \rightarrow \text{vrai}$$

Supposons $U_n = \sqrt{e} - I_n$ vrai.

$$\text{Df } U_{n+1} = \sqrt{e} - I_{n+1}$$

$$\text{en effet: } U_{n+1} = U_n + \frac{1}{(n+1)! \cdot 2^{n+1}} = U_n + I_n - I_{n+1}$$

$$\Rightarrow U_{n+1} = \sqrt{e} - I_n + I_n - I_{n+1} = \sqrt{e} - I_{n+1}$$

$$c) 0 \leq f(x) \leq 1 \Leftrightarrow 0 \leq x e^{\frac{1-x}{2}} \leq 1$$

$$x^{n-1} \geq 0 \Rightarrow 0 \leq x^n e^{\frac{1-x}{2}} \leq x^{n-1}$$

$$\Rightarrow 0 \leq \int_0^1 x^n e^{\frac{1-x}{2}} dx \leq \int_0^1 x^{n-1} dx = \frac{1}{n}$$

$$\Rightarrow 0 \leq I_n \leq \frac{1}{n(n!) \cdot 2^{n+1}} \Rightarrow \lim_{n \rightarrow \infty} I_n = 0$$

Problème 18 : BAC 1996 - Session Principale

A) $f(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x} + \frac{1}{4x^2}$

1) f existe si $\begin{cases} x \neq 0 \\ 1 + \frac{1}{x} > 0 \Rightarrow \frac{x+1}{x} > 0 \end{cases}$

x	$-\infty$	-1	0	$+\infty$
$x+1$	$-$	0	$+$	$+$
x	$-$	$-$	0	$+$
$\frac{x+1}{x}$	$+$	0	$-$	$+$

$\Rightarrow D_f =]-\infty, -1[\cup]0, +\infty[$

$f \in \mathcal{C}^1$ sur $D_f =]-\infty, -1[\cup]0, +\infty[$

$$f'(x) = \frac{-\frac{1}{x^2}}{1 + \frac{1}{x}} + \frac{1}{x^2} - \frac{2}{4x^3} = \frac{-1}{x(x+1)} + \frac{1}{x^2} - \frac{1}{2x^3}$$

$$= \frac{-2x^2 + 2x(x+1) - (x+1)}{2x^2 \cdot x(x+1)} = \frac{x-1}{2x^2 \cdot x(x+1)}$$

$\text{sgn}(f') = \text{sgn}(x-1)$ car $x(x+1) > 0$

x	$-\infty$	-1	0	1	$+\infty$
$f'(x)$	$-$			$-$	$+$
$f(x)$	0			0	0

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x} + \frac{1}{4x^2} \right] = 0$$

$\Rightarrow y=0$ Asymptote horizontale en $(\pm\infty)$

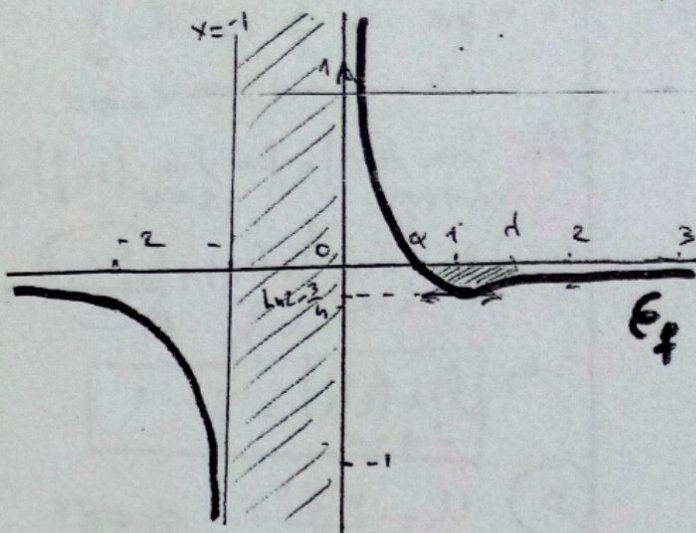
$$\lim_{x \rightarrow (-1)^-} f(x) = \lim_{x \rightarrow (-1)^-} \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x} + \frac{1}{4x^2} \right] = -\infty$$

$\Rightarrow x=-1$ Asymptote verticale

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x} + \frac{1}{4x^2} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\ln(1+x) - \ln x - \frac{1}{x} + \frac{1}{4x^2} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\ln(1+x) - \frac{4x^2 \ln x + 4x - 1}{4x^2} \right] = +\infty \end{aligned}$$

$\Rightarrow x=0$ Asymptote verticale à la courbe $\mathcal{C}_f$

$$f(1) = \ln 2 - 1 + \frac{1}{4} = \ln 2 - \frac{3}{4}$$



2) a) $f \in \mathcal{C}^1$ sur $]0, 1[\Rightarrow f$ réalise une bijection de $]0, 1[$ sur $[\ln 2 - \frac{3}{4}, +\infty[$ car $0 \in [\ln 2 - \frac{3}{4}, +\infty[\Rightarrow$ il existe un unique $\alpha \in]0, 1[$ / $f(\alpha) = 0$

b) on a $\forall x \geq \alpha$, $\mathcal{C}_f$ est située au-dessous de l'axe (Ox)

$$\Rightarrow \forall x \geq \alpha, \boxed{f(x) \leq 0}$$

$$3) A(d) = \int_{\alpha}^d f(x) dx = \int_{\alpha}^d \left[\ln\left(1 + \frac{1}{x}\right) + \frac{1}{x} - \frac{1}{4x^2} \right] dx$$

$$\Rightarrow A(d) = \left[\ln x + \frac{1}{4x} \right]_{\alpha}^d - \int_{\alpha}^d \ln\left(1 + \frac{1}{x}\right) dx$$

$$\text{on pose } u(x) = \ln\left(1 + \frac{1}{x}\right) \rightarrow u'(x) = \frac{-\frac{1}{x^2}}{1 + \frac{1}{x}} = -\frac{1}{x(x+1)}$$

$$v'(x) = 1 \rightarrow v(x) = x$$

$$\Rightarrow I = \left[x \ln\left(1 + \frac{1}{x}\right) \right]_{\alpha}^d + \int_{\alpha}^d \frac{1}{x+1} dx$$

$$= \left[x \ln\left(1 + \frac{1}{x}\right) + \ln(1+x) \right]_{\alpha}^d$$

$$\Rightarrow A(d) = \left[\ln x + \frac{1}{4x} - x \ln\left(1 + \frac{1}{x}\right) - \ln(1+x) \right]_{\alpha}^d$$

$$\Rightarrow A(d) = \left[(-1-x) \ln\left(1 + \frac{1}{x}\right) + \frac{1}{4x} \right]_{\alpha}^d$$

$$= (-1-d) \ln\left(1 + \frac{1}{d}\right) + \frac{1}{4d} + (1+\alpha) \ln\left(1 + \frac{1}{\alpha}\right) - \frac{1}{4\alpha}$$

$$\Rightarrow \boxed{A(d) = (1+\alpha) \ln\left(1 + \frac{1}{\alpha}\right) - \frac{1}{4\alpha} - (1+d) \ln\left(1 + \frac{1}{d}\right) + \frac{1}{4d}}$$

$$\text{on pose } \beta = (1+\alpha) \ln\left(1 + \frac{1}{\alpha}\right) - \frac{1}{4\alpha} : \text{cte}$$

$$\Rightarrow \boxed{A(d) = \beta - (1+d) \ln\left(1 + \frac{1}{d}\right) + \frac{1}{4d}} \quad (\text{ka})$$

• on pose $x = \frac{1}{d} \Rightarrow d = \frac{1}{x}$
 si $d \rightarrow +\infty \Rightarrow x \rightarrow 0^+$

d'où $\lim_{d \rightarrow +\infty} A(d) = \beta - \lim_{x \rightarrow 0^+} \left(\left(1 + \frac{1}{x}\right) \ln(1+x) - \frac{1}{4}x \right)$
 $= \beta - \lim_{x \rightarrow 0^+} \left[\underbrace{\ln(1+x)}_{\rightarrow 0} + \underbrace{\ln\left(\frac{1+x}{x}\right)}_{\rightarrow 1} - \frac{1}{4}x \right]$
 $\Rightarrow \boxed{\lim_{d \rightarrow +\infty} A(d) = \beta - 1}$

(B) $U_n = \frac{n^n e^{-n}}{n!} \quad \forall n \in \mathbb{N}^*$

1) on a $\forall x \geq \alpha, f(x) \leq 0$
 or $n \in \mathbb{N}^* \Rightarrow n \geq \alpha$ d'où $f(n) \leq 0$

$$\Rightarrow \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n} + \frac{1}{4n^2} \leq 0$$

$$\Rightarrow \ln\left(\frac{n+1}{n}\right) \leq \frac{1}{n} - \frac{1}{4n^2}$$

$$\Rightarrow \boxed{\ln(n+1) - \ln n \leq \frac{1}{n} - \frac{1}{4n^2}}$$

$$\cdot \left(\ln\left(\frac{n+1}{n}\right) \leq \frac{1}{n} - \frac{1}{4n^2} \right) \cdot n$$

$$\Rightarrow \ln\left[\left(\frac{n+1}{n}\right)^n\right] \leq 1 - \frac{1}{4n}$$

$$e^{\uparrow} \Rightarrow \boxed{\left(\frac{n+1}{n}\right)^n \leq e^{1 - \frac{1}{4n}}} \quad \forall n \in \mathbb{N}^*$$

2) on a $\forall n \in \mathbb{N}^*, U_n > 0$

$$\frac{U_{n+1}}{U_n} = \frac{\frac{(n+1)^{n+1} e^{-(n+1)}}{(n+1)!}}{\frac{n^n e^{-n}}{n!}} = \frac{(n+1)^{n+1} e^{-n-1}}{(n+1)!} \cdot \frac{n!}{n^n e^{-n}}$$

$$= \frac{(n+1) \cdot (n+1)^n \cdot e^{-1} \cdot e^{-n}}{(n+1) \cdot n!} \cdot \frac{n!}{n^n e^{-n}}$$

$$\frac{U_{n+1}}{U_n} = \left(\frac{n+1}{n}\right)^n e^{-1} \Rightarrow \frac{U_{n+1}}{U_n} = \left(\frac{n+1}{n}\right)^n e^{-1}$$

$$\text{or } \left(\frac{n+1}{n}\right)^n \leq e^{1 - \frac{1}{4n}} \Rightarrow \left(\frac{n+1}{n}\right)^n e^{-1} \leq e^{-\frac{1}{4n}}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} \leq e^{-\frac{1}{4n}} \Rightarrow \boxed{U_{n+1} \leq U_n e^{-\frac{1}{4n}}}$$

• Or $\forall n \in \mathbb{N}^*, U_{n+1} \leq U_n e^{-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}}$

Pour $n=1$: $U_2 \leq U_1 e^{-\frac{1}{4}}$ vérifie

suppose $U_n \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}}$ va

Or $U_{n+1} \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^{n+1} \frac{1}{k^2}}$?

en effet: $U_n \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}}$

$$\text{et } U_{n+1} \leq U_n e^{-\frac{1}{4n}}$$

Produit: $U_{n+1} \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^{n+1} \frac{1}{k^2}}$

$$\Rightarrow U_{n+1} \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^{n+1} \frac{1}{k^2}}$$

Cad: $\forall n \in \mathbb{N}^* \quad \boxed{U_{n+1} \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^{n+1} \frac{1}{k^2}}}$

3) on a $\ln(k+1) - \ln k \leq \frac{1}{k} - \frac{1}{4k^2}$

d'où $\sum_{k=1}^n \ln(k+1) - \sum_{k=1}^n \ln k \leq \sum_{k=1}^n \frac{1}{k} - \frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}$

$$\Rightarrow \boxed{\ln(n+1) \leq \sum_{k=1}^n \frac{1}{k} - \frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}}$$

• Comme $-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2} \leq 0$

$$\Rightarrow \ln(n+1) \leq \sum_{k=1}^n \frac{1}{k} - \frac{1}{4} \sum_{k=1}^n \frac{1}{k^2} \leq \sum_{k=1}^n \frac{1}{k}$$

d'où $-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2} \leq -\frac{1}{4} \ln(n+1)$

$$e^{\uparrow} \Rightarrow e^{-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}} \leq e^{-\frac{1}{4} \ln(n+1)}$$

$$U_n > 0 \Rightarrow U_1 e^{-\frac{1}{4} \sum_{k=1}^n \frac{1}{k^2}} \leq U_1 e^{-\frac{1}{4} \ln(n+1)}$$

$$\text{or } U_{n+1} \leq U_1 e^{-\frac{1}{4} \sum_{k=1}^{n+1} \frac{1}{k^2}} \leq U_1 e^{-\frac{1}{4} \ln(n+1)}$$

alors $\forall n \in \mathbb{N}^* \quad \boxed{U_{n+1} \leq U_1 e^{-\frac{1}{4} \ln(n+1)}}$

4) comme $U_n > 0$ et $U_n \leq U_1 e^{-\frac{1}{4} \ln(n+1)}$

d'où $\forall n \in \mathbb{N}^*, 0 < U_n \leq U_1 e^{-\frac{1}{4} \ln(n+1)}$

$$U_1 = \frac{1 \cdot e^{-1}}{1!} = e^{-1}$$

$$\Rightarrow 0 < U_n \leq e^{-1 - \frac{1}{4} \ln n}$$

$$\lim_{n \rightarrow +\infty} (e^{-1 - \frac{1}{4} \ln n}) = 0$$

$$\Rightarrow \boxed{\lim U_n = 0}$$

on a: $\lim U_n$ fini $\Rightarrow$

U_n est convergente.

Problème 19 : BAC 1996 - Session de Contrôle

1) $f_n(x) = (x-1)^n \cdot \ln x$ $D_{f_n} =]0, +\infty[$

1) $\varphi_n(x) = n \ln x + 1 - \frac{1}{x}$ $D_{\varphi} = \mathbb{R}_+^*$

a) $\varphi_n \in \mathcal{C}^1$ sur $]0, +\infty[$; $\varphi'_n(x) = \frac{n}{x} + \frac{1}{x^2} > 0$

x	0	1	$+\infty$
φ'_n		+	
φ	$-\infty$		$+\infty$

$$\lim_{x \rightarrow 0^+} \varphi_n = \lim_{x \rightarrow 0^+} n \ln x + 1 - \frac{1}{x} = -\infty$$

$$\lim_{x \rightarrow +\infty} \varphi_n = \lim_{x \rightarrow +\infty} n \ln x + 1 - \frac{1}{x} = +\infty$$

b) $\varphi_n(1) = 0 + 1 - 1 = 0$

donc si $x \in]0, 1[$, $\varphi_n(x) \in]-\infty, 0] \Rightarrow \varphi_n \leq 0$

si $x \in [1, +\infty[$, $\varphi_n(x) \in [0, +\infty[\Rightarrow \varphi_n(x) \geq 0$

Caract: $\begin{array}{c|ccc} x & 0 & 1 & +\infty \\ \hline \varphi_n(x) & - & 0 & + \end{array}$

2) a) f_n continue et dérivable sur $]0, +\infty[$

$$f'_n(x) = n(x-1)^{n-1} \ln x + \frac{1}{x} (x-1)^n$$

$$\Rightarrow f'_n(x) = (x-1)^{n-1} \left[n \ln x + 1 \right]$$

$$\Rightarrow f'_n(x) = (x-1)^{n-1} \varphi_n(x)$$

• Si $(n-1)$: pair $\Leftrightarrow n$: impair $\Rightarrow (x-1)^{n-1} \geq 0$
 alors le signe $[f'_n(x)] =$ signe de $\varphi_n(x)$

x	0	1	$+\infty$
$f'_n(x)$	-	0	+
$f_n(x)$	$-\infty$		$+\infty$

• Si $(n-1)$: impair $\Leftrightarrow n$: pair

x	0	1	$+\infty$
$(x-1)^{n-1}$	-	0	+
$\varphi_n(x)$	-	0	+
$f'_n(x)$	+	0	+
$f_n(x)$	$-\infty$		$+\infty$

$$\lim_{x \rightarrow +\infty} f_n(x) = \lim_{x \rightarrow +\infty} (x-1)^n \ln x = +\infty \quad \forall n \in \mathbb{N}^*$$

$$\lim_{x \rightarrow 0^+} f_n(x) = \lim_{x \rightarrow 0^+} (x-1)^n \ln x = (-1)^n (-\infty)$$

• Si n : pair $\Rightarrow f_n = -\infty$

• Si n : impair $\Rightarrow f_n = +\infty$

• $f_n(1) = 0$

b) Position de $\mathcal{C}_1$ et $\mathcal{C}_2$:

Soit $g(x) = f_1(x) - f_2(x) = (x-1) \ln x - (x-1)^2 \ln x$

$$\Rightarrow g(x) = (x-1) \ln x \cdot (1-x+1)$$

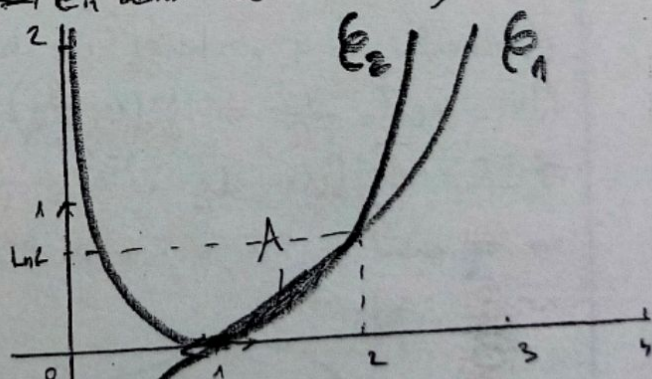
$$\Rightarrow g(x) = (x-1)(2-x) \ln x$$

$$g(x) = 0 \Leftrightarrow \begin{cases} x-1=0 \Leftrightarrow x=1 \\ 2-x=0 \Leftrightarrow x=2 \\ \ln x=0 \Leftrightarrow x=1 \end{cases}$$

x	0	1	2	$+\infty$
$x-1$	-	0	+	+
$2-x$	+	+	0	-
$\ln x$	-	0	+	+
$g(x)$	+	0	+	0

Position de $\mathcal{C}_1$ et $\mathcal{C}_2$	$\frac{\mathcal{C}_1}{\mathcal{C}_2}$	$\frac{\mathcal{C}_1}{\mathcal{C}_2}$	$\frac{\mathcal{C}_1}{\mathcal{C}_2}$	$\frac{\mathcal{C}_2}{\mathcal{C}_1}$
--	---------------------------------------	---------------------------------------	---------------------------------------	---------------------------------------

B.I.: $\lim_{x \rightarrow +\infty} \frac{f_1(x)}{f_2(x)} = \lim_{x \rightarrow +\infty} \frac{(x-1) \ln x}{(x-1)^2 \ln x} = \lim_{x \rightarrow +\infty} \frac{1}{x-1} = 0$
 car $\lim_{x \rightarrow +\infty} \ln x = +\infty$ et $\forall n \in \mathbb{N}^*$, $\lim_{x \rightarrow +\infty} \frac{(x-1)^n}{x} = +\infty$ si $n=1$
 $\Rightarrow$ En admettant une B.I. en $(+\infty)$ de $\ln(0^+)$



$$3) A = \int_1^2 (f_1(x) - f_2(x)) dx = \int_1^2 g(x) dx$$

$$\Rightarrow A = \int_1^2 (-x^2 + 3x - 2) \ln x dx$$

on pose $| u(x) = \ln x \longrightarrow u'(x) = \frac{1}{x}$
 $| v'(x) = -x^2 + 3x - 2 \longrightarrow v(x) = -\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x$

$$\Rightarrow A = \left[\left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x \right) \ln x \right]_1^2 - \int_1^2 \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x \right) dx$$

$$\Rightarrow A = \left[\left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x \right) \ln x + \frac{1}{9}x^3 - \frac{3}{4}x^2 + 2x \right]_1^2$$

$$= \left[\left(-\frac{8}{3} + 6 - 4 \right) \ln 2 + \frac{8}{9} - 3 + 4 \right] - \left[0 + \frac{1}{9} - \frac{3}{4} + 2 \right]$$

$$= -\frac{2}{3} \ln 2 + \frac{17}{9} - \frac{49}{36}$$

$$\Rightarrow \boxed{A = \frac{19}{36} - \frac{2}{3} \ln 2} \text{ (ua)}$$

(B) $V_n = \sum_{k=0}^n \frac{(-1)^k}{k+1}$

1) $U_n = \int_1^2 f_n(x) dx$

a) $(n+1)U_n = \int_1^2 (n+1)(x-1)^n \cdot \ln x dx$

on pose $| h(x) = \ln x \longrightarrow h'(x) = \frac{1}{x}$
 $| k'(x) = (n+1)(x-1)^n \longrightarrow k(x) = (x-1)^{n+1}$

$$\Rightarrow (n+1)U_n = \left[(x-1)^{n+1} \ln x \right]_1^2 - \int_1^2 \frac{(x-1)^{n+1}}{x} dx$$

$$\Rightarrow (n+1)U_n = \ln 2 - \int_1^2 \frac{(x-1)^{n+1}}{x} dx$$

b) $\int_1^2 \frac{(x-1)^{n+1}}{x} dx = \ln 2 - (n+1)U_n$

or $1 \leq x \leq 2 \Rightarrow \frac{1}{2} \leq \frac{1}{x} \leq 1$

or $(x-1)^{n+1} \geq 0 \Rightarrow \frac{1}{2}(x-1)^{n+1} \leq \frac{(x-1)^{n+1}}{x} \leq (x-1)^{n+1}$

Comme $1 < 2 \Rightarrow \int_1^2 \frac{1}{2}(x-1)^{n+1} dx \leq \int_1^2 \frac{(x-1)^{n+1}}{x} dx \leq \int_1^2 (x-1)^{n+1} dx$

$$\Rightarrow \left[\frac{1}{2} \cdot \frac{(x-1)^{n+2}}{n+2} \right]_1^2 \leq \ln 2 - (n+1)U_n \leq \left[\frac{(x-1)^{n+2}}{n+2} \right]_1^2$$

$$\Rightarrow \boxed{\frac{1}{2(n+2)} \leq \ln 2 - (n+1)U_n \leq \frac{1}{n+2}}$$

c) $\lim_{n \rightarrow \infty} \frac{1}{2(n+2)} = 0$
 $\lim_{n \rightarrow \infty} \frac{1}{n+2} = 0 \Rightarrow \lim_{n \rightarrow \infty} (\ln 2 - (n+1)U_n) = 0$
 $\Rightarrow \ln 2 - \lim_{n \rightarrow \infty} (n+1)U_n = 0$

$$\Rightarrow \boxed{\lim_{n \rightarrow \infty} (n+1)U_n = \ln 2}$$

2) $S_n(x) = 1 - (x-1) + (x-1)^2 - \dots + (-1)^n (x-1)^n$

a) si $x = 1 \Rightarrow S_n(1) = 1$ vérifie

si $x \neq 1$: on considère la suite géométrique w_n

définie par $w_0 = 1$ et $q = -(x-1)$

$$\Rightarrow S_n(x) = \sum_{k=0}^n w_k = w_0 \left[\frac{1 - q^{n+1}}{1 - q} \right]$$

$$\Rightarrow S_n(x) = \frac{1 - [-(x-1)]^{n+1}}{1 + (x-1)} = \frac{1 - (-1)^{n+1}(x-1)^{n+1}}{x}$$

$$\Rightarrow S_n(x) = \frac{1}{x} - (-1)^{n+1} \frac{(x-1)^{n+1}}{x}$$

car: $\forall x > 0$ et $\forall n \in \mathbb{N}^*$ on a:

$$\boxed{S_n(x) = \frac{1}{x} - (-1)^{n+1} \frac{(x-1)^{n+1}}{x}}$$

b) or $S_n(x) = \sum_{k=0}^n (-1)^k (x-1)^k$

d'où $\sum_{k=0}^n (-1)^k (x-1)^k = \frac{1}{x} - (-1)^{n+1} \frac{(x-1)^{n+1}}{x}$

d'où $\int_1^2 \sum_{k=0}^n (-1)^k (x-1)^k dx = \int_1^2 \frac{1}{x} dx - \int_1^2 (-1)^{n+1} \frac{(x-1)^{n+1}}{x} dx$

$$\Rightarrow \sum_{k=0}^n (-1)^k \int_1^2 (x-1)^k dx = \left[\ln x \right]_1^2 + (-1)^{n+1} \int_1^2 \frac{(x-1)^{n+1}}{x} dx$$

$$\Rightarrow \sum_{k=0}^n (-1)^k \left[\frac{(x-1)^{k+1}}{k+1} \right]_1^2 = \ln 2 + (-1)^{n+1} \left[\ln 2 - (n+1)U_n \right]$$

$$\Rightarrow \sum_{k=0}^n \frac{(-1)^k}{k+1} = \ln 2 + (-1)^{n+1} \left[\ln 2 - (n+1)U_n \right]$$

$$\Rightarrow V_n = \ln 2 + (-1)^{n+1} \left[\ln 2 - (n+1)U_n \right]$$

$$\Rightarrow \ln 2 - V_n = -(-1)^{n+1} \left[\ln 2 - (n+1)U_n \right]$$

$$\Rightarrow \boxed{\ln 2 - V_n = (-1)^{n+1} \left[\ln 2 - (n+1)U_n \right]}$$

$$\Rightarrow |\ln 2 - V_n| = |(-1)^{n+1}| \cdot |\ln 2 - (n+1)U_n|$$

$$\Rightarrow |\ln 2 - V_n| = |\ln 2 - (n+1)U_n|$$

or $\lim_{n \rightarrow \infty} (\ln 2 - (n+1)U_n) = 0$

$$\Rightarrow \lim_{n \rightarrow \infty} \ln 2 - V_n = 0 \Rightarrow \lim_{n \rightarrow \infty} V_n = \ln 2$$

car: $\boxed{V_n \text{ converge vers } \ln 2}$

Problème 20 : BAC 1995 - Session de Contrôle

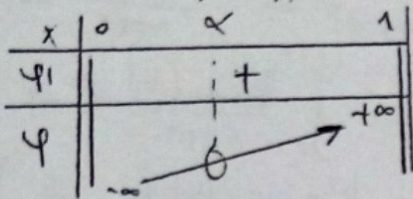
$$I) f(x) = \frac{-x \ln x}{1+x^2} \quad x \in]0,1]; f(0)=0$$

$$2) \varphi(x) = \ln x + \frac{1+x^2}{1-x^2} \quad D\varphi =]0,1[$$

$$a) \varphi \in \mathcal{C}^1 \text{ sur }]0,1[\text{ car } x > 0 \text{ et } 1-x^2 \neq 0$$

$$\varphi'(x) = \frac{1}{x} + \frac{2x(1-x^2) + 2x(1+x^2)}{(1-x^2)^2} = \frac{1}{x} + \frac{4x}{(1-x^2)^2}$$

$$\Rightarrow \forall x \in]0,1[, \varphi'(x) > 0$$

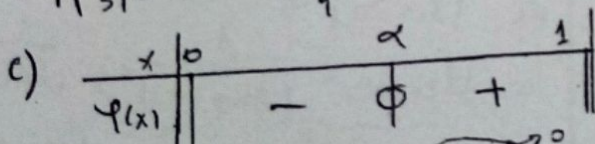


$$\lim_{0^+} \varphi(x) = \lim_{0^+} \left(\ln x + \frac{1+x^2}{1-x^2} \right) = -\infty$$

$$\lim_{1^-} \varphi(x) = \lim_{1^-} \left(\ln x + \frac{1+x^2}{1-x^2} \right) = +\infty$$

b) $\varphi \in \mathcal{C}^1$ sur $]0,1[\Rightarrow \varphi$ réalise une bijection de $]0,1[$ sur $\mathbb{R}$
comme $0 \in \mathbb{R} \Rightarrow \exists ! \alpha \in]0,1[/ \varphi(\alpha) = 0$

$$\begin{cases} \varphi(\frac{1}{4}) = -\ln 4 + \frac{17}{15} < 0 \\ \varphi(\frac{1}{3}) = -\ln 3 + \frac{5}{4} > 0 \end{cases} \Rightarrow \boxed{\frac{1}{4} < \alpha < \frac{1}{3}}$$



$$2) a) \lim_{0^+} f(x) = \lim_{0^+} \frac{-x \ln x}{1+x^2} = 0 = f(0)$$

$\Rightarrow f$ est continue en 0

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{0^+} \frac{-x \ln x}{1+x^2}$$

$$= \lim_{0^+} \frac{-\ln x}{1+x^2} = +\infty \Rightarrow f \text{ n'est pas dérivable en } 0$$

seulement en 0 et $\mathcal{E}_f$ admet en $0(0,0)$ une demi-tangente verticale dirigée vers le haut.

$$b) \forall x \in]0,1[, \ln x : \frac{1}{x} \text{ et } 1+x^2 \neq 0$$

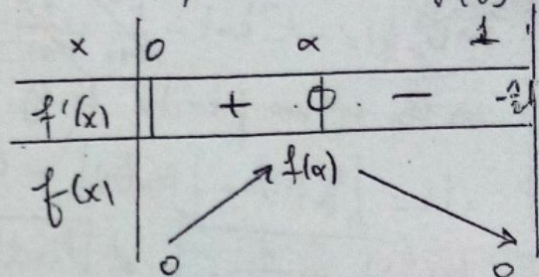
$\Rightarrow f$ est dérivable sur $]0,1[$

$$f'(x) = \frac{-(\ln x + 1)(1+x^2) + 2x^2 \ln x}{(1+x^2)^2}$$

$$\Leftrightarrow f'(x) = \frac{-\ln x - x^2 \ln x - (1+x^2) + 2x^2 \ln x}{(1+x^2)^2}$$

$$\Leftrightarrow f'(x) = \frac{(x^2-1)(\ln x + \frac{1+x^2}{1-x^2})}{(1+x^2)^2} = \frac{(x-1)\varphi(x)}{(1+x^2)^2}$$

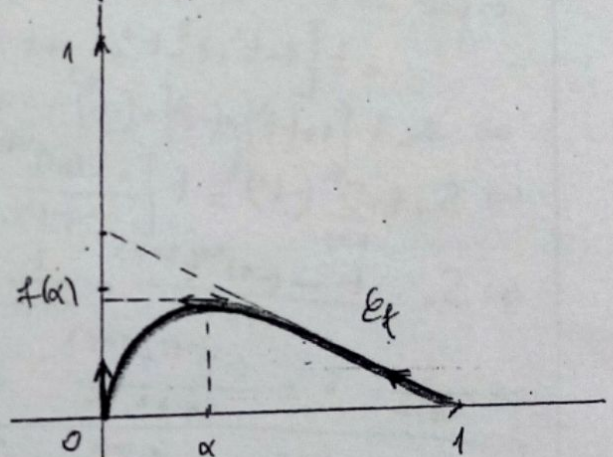
$$c) \forall x \in]0,1[, (x^2-1) < 0 \Rightarrow \text{sgn}(f') = -\text{sgn}(\varphi)$$



$$T_1: y = f'(1)(x-1) + f(1) \Rightarrow y = -\frac{1}{2}x + \frac{1}{2}$$

$$\text{on } f'(1) = -\frac{1}{2} \text{ et } f(1) = 0$$

$$\text{pm } \mathcal{E}_f; \text{ on prend } \alpha \geq 0, \beta \Rightarrow f(\alpha) \geq 0, \beta$$



$$II) I = \int_0^1 f(x) dx$$

$$1) g_n(t) = -t^n \ln t \quad \text{si } t > 0 \text{ et } g_n(0) = 0$$

$$a) g_n \text{ intégrable sur }]0,1[\Leftrightarrow g_n \in \mathcal{M}[0,1]$$

$$\text{or } g_n \in \mathcal{M}[0,1] \text{ équivaut à } g_n \in \mathcal{C}^1 \text{ en } 0$$

$$\lim_{0^+} g_n(t) = \lim_{0^+} -t^n \ln t = 0 \Rightarrow g_n \in \mathcal{C}^1 \text{ en } 0$$

$$\text{alors } g_n \in \mathcal{M}[0,1] \Rightarrow g_n \text{ intégrable sur }]0,1[$$

$$I_n = \int_0^1 g_n(t) dt$$

$$b) G_n(t) = -\frac{t^{n+1}}{n+1} \ln t + \frac{t^{n+1}}{(n+1)^2} \quad \text{et } G_n(0) = 0$$

$$\text{Or } G_n \text{ primitive de } g_n \text{ sur }]0,1[$$

$$\text{or } G_n \in \mathcal{M}[0,1] \text{ si } t > 0 \Rightarrow G_n \in \mathcal{M}[0,1]$$

$$\text{évaluons alors la } \int_0^1 \text{ de } G_n \text{ en } 0:$$

$$\lim_{t \rightarrow 0^+} \frac{G_n(t) - G_n(0)}{t - 0} = \lim_{t \rightarrow 0^+} \frac{-t^{n+1} \ln t + \frac{t^{n+1}}{(n+1)^2}}{t}$$

$$= \lim_{t \rightarrow 0^+} -\frac{t^n \ln t}{n+1} + \frac{t^n}{(n+1)^2} = 0 \Rightarrow G_n \text{ est en } 0^+$$

$$G'_n(t) = -\frac{1}{n+1} \left[(n+1)t^n \ln t + t^{n+1} \left(\frac{1}{t} \right) \right] + \frac{(n+1)t^n}{(n+1)^2}$$

$$\Rightarrow G'_n(t) = -t^n \ln t - \frac{t^n}{n+1} + \frac{t^n}{n+1} = g_n(t)$$

$$\Rightarrow G_n \text{ est une primitive de } g_n \text{ sur }]0, 1[$$

$$U_n = \int_0^1 g_n(t) dt = [G_n(t)]_0^1 = G_n(1) - G_n(0)$$

$$\Rightarrow U_n = \frac{1}{(n+1)^2} \quad \text{et} \quad \lim U_n = 0$$

$$a) \text{ on a } \frac{t}{1+t^2} = t - t^3 + t^5 - \dots + (-1)^n t^{2n+1} + (-1)^{n+1} \frac{t^{2n+3}}{1+t^2}$$

$$\text{on pose } S = t - t^3 + t^5 - \dots + (-1)^n t^{2n+1}$$

$$= t [1 - t^2 + t^4 - \dots + (-1)^n t^{2n}]$$

$$\Rightarrow S = t [1 + (-t^2) + (-t^2)^2 + \dots + (-t^2)^n]$$

$$\Rightarrow S = t \sum_{k=0}^n (-t^2)^k = t \left[\frac{1 - (-t^2)^{n+1}}{1 - (-t^2)} \right] = t \left[\frac{1 - (-1)^{n+1} t^{2n+2}}{1 + t^2} \right]$$

$$\Rightarrow S = \frac{t - (-1)^{n+1} t^{2n+3}}{1+t^2} = \frac{t}{1+t^2} - \frac{(-1)^{n+1} t^{2n+3}}{1+t^2}$$

$$\Rightarrow \frac{t}{1+t^2} = S + \frac{(-1)^{n+1} t^{2n+3}}{1+t^2} \quad \text{d'où}$$

$$\frac{t}{1+t^2} = t - t^3 + t^5 - \dots + (-1)^n t^{2n+1} + \frac{(-1)^{n+1} t^{2n+3}}{1+t^2}$$

$$b) \frac{-t \ln t}{1+t^2} = -t \ln t + t^3 \ln t - t^5 \ln t + \dots + (-1)^n t^{2n+1} \ln t - \frac{(-1)^{n+1} t^{2n+3} \ln t}{1+t^2}$$

$$\Rightarrow f(t) = g_1(t) - g_3(t) + \dots + (-1)^n g_{2n+1}(t) + \frac{(-1)^{n+1} g_{2n+3}(t)}{1+t^2}$$

on intègre l'expression précédente :

$$\Rightarrow \int_0^1 f(t) dt = \int_0^1 g_1(t) dt - \int_0^1 g_3(t) dt + \dots$$

$$+ (-1)^n \int_0^1 g_{2n+1}(t) dt + (-1)^{n+1} \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt$$

$$\Rightarrow I = U_1 - U_3 + U_5 - \dots + (-1)^n U_{2n+1} + (-1)^{n+1} \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt$$

$$3) S_n = U_1 - U_3 + U_5 - \dots + (-1)^n U_{2n+1}$$

$$a) I = S_n + (-1)^{n+1} \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt$$

$$\Rightarrow I - S_n = (-1)^{n+1} \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt$$

$$\text{et } |(-1)^{n+1}| = 1 \quad \text{et } g_{2n+3}(t) = -t^{2n+3} \ln t > 0$$

$$\Rightarrow |I - S_n| = \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt$$

$$\text{or } 0 \leq t \leq 1 \Rightarrow 1 \leq 1+t^2 \leq 2$$

$$\Rightarrow \frac{1}{1+t^2} \leq 1$$

$$g_{2n+3} \geq 0 \Rightarrow \frac{g_{2n+3}(t)}{1+t^2} \leq g_{2n+3}(t)$$

$$\text{d'où } \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt \leq \int_0^1 g_{2n+3}(t) dt$$

$$\Rightarrow \int_0^1 \frac{g_{2n+3}(t)}{1+t^2} dt \leq U_{2n+3}$$

$$\text{alors } |I - S_n| \leq U_{2n+3}$$

$$b) \lim U_n = 0 \Rightarrow \lim U_{2n+3} = 0$$

$$\text{d'où } \lim (I - S_n) = 0$$

$$\Rightarrow \boxed{\lim S_n = I}$$

$$c) |I - S_n| \leq 10^{-2}$$

$$\text{or } |I - S_n| \leq U_{2n+3} \leq 10^{-2}$$

$$\text{or } U_{2n+3} = \frac{1}{(2n+3+1)^2} = \frac{1}{(2n+4)^2} \leq 10^{-2}$$

$$\Rightarrow (2n+4)^2 \geq 10^2 \quad \text{d'où } 2n+4 \geq 10$$

$$\Rightarrow 2n \geq 6 \Rightarrow n \geq 3$$

$$\text{alors } \boxed{n_0 = 3}$$

$$S_3 = U_1 - U_3 + U_5 - U_7$$

$$\text{or } U_1 = \frac{1}{(1+1)^2} = \frac{1}{4}, \quad U_3 = \frac{1}{(3+1)^2} = \frac{1}{16}$$

$$U_5 = \frac{1}{(5+1)^2} = \frac{1}{36}, \quad U_7 = \frac{1}{(7+1)^2} = \frac{1}{64}$$

$$\Rightarrow S_3 = \frac{1}{4} - \frac{1}{16} + \frac{1}{36} - \frac{1}{64} = \frac{115}{576}$$

$$\text{d'où } \boxed{I \approx \frac{115}{576}} \text{ à } 10^{-2} \text{ près}$$

Problème 21 : BAC 1993 - Session Principale

(A) $f_m(x) = m \cdot x \cdot \ln|x| - x$ si $x \neq 0$
 $f_m(0) = 0$

1) a) $\subseteq$ de f_m en 0 :

$$\lim_{x \rightarrow 0^+} f_m(x) = \lim_{x \rightarrow 0^+} m \cdot x \cdot \ln(x) - x = 0 = f_m(0)$$

$$\lim_{x \rightarrow 0^-} f_m(x) = \lim_{x \rightarrow 0^-} m \cdot x \cdot \ln(-x) - x$$

$$= \lim_{t \rightarrow 0^+} -m(t \cdot \ln t) + t = 0 = f_m(0)$$

$$\Rightarrow \lim_{x \rightarrow 0} f_m(x) = \lim_{x \rightarrow 0} f_m(x) = f_m(0) = 0$$

$$\Rightarrow f_m \subseteq \text{en } 0 \quad \forall m \in \mathbb{R}^*$$

$\subseteq$ de f en 0 :

$$\lim_{x \rightarrow 0} \frac{f_m(x) - f_m(0)}{x - 0} = \lim_{x \rightarrow 0} (m \ln|x| - 1) = \pm \infty$$

$\Rightarrow f_m$ n'est pas $\subseteq$ en 0 et E_m admet une tangente verticale en 0, $\forall m \in \mathbb{R}^*$

b) on a $\forall x \in \mathbb{R}, (-x) \in \mathbb{R}$

$$f_m(-x) = m(-x) \cdot \ln|-x| + x = -m \cdot x \cdot \ln|x| + x$$

$$= -[m \cdot x \cdot \ln|x| - x] = -f_m(x)$$

$\Rightarrow f_m$ est une fonction impaire

$$\cdot \forall x \in \mathbb{R}^*. f'_m(x) = m \ln|x| + m \cdot x \cdot \frac{1}{x} - 1$$

$$\Rightarrow f'_m(x) = m \ln|x| + m - 1$$

comme f_m impaire $\Rightarrow D = \mathbb{R}^+$

$$\Rightarrow f'_m(x) = m \cdot \ln x + m - 1$$

$$\text{on résout : } f'_m(x) = 0 \Leftrightarrow m \ln x + m - 1 = 0$$

$$\Leftrightarrow m \cdot \ln x = 1 - m \Leftrightarrow \ln x = \frac{1}{m} - 1$$

$$\Leftrightarrow x = e^{\frac{1}{m} - 1}$$

• si $m > 0$

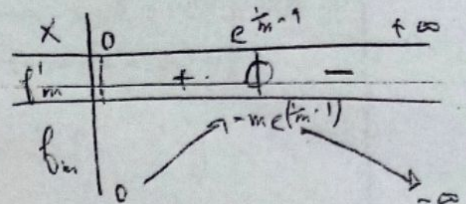
$$f_m(e^{\frac{1}{m}-1}) = m \cdot e^{\frac{1}{m}-1} \cdot \ln(e^{\frac{1}{m}-1}) - e^{\frac{1}{m}-1}$$

$$= (1-m) e^{\frac{1}{m}-1} - e^{\frac{1}{m}-1}$$

$$\Rightarrow f_m(e^{\frac{1}{m}-1}) = -m e^{\frac{1}{m}-1}$$

$$\lim_{x \rightarrow +\infty} f_m(x) = \lim_{x \rightarrow +\infty} x(m \ln x - 1) = +\infty \quad (B.J.)$$

• si $m < 0$



$$\lim_{x \rightarrow +\infty} f_m(x) = \lim_{x \rightarrow +\infty} x(m \ln x - 1) = -\infty$$

$$f_m(e^{\frac{1}{m}-1}) = -m e^{\frac{1}{m}-1}$$

c) A_m et B_m extrêmes de E_m

comme f_m impaire $\Rightarrow B_m = S_0(A_m)$

$$\text{et } A_m(e^{\frac{1}{m}-1}, -m e^{\frac{1}{m}-1})$$

Ensemble des points A_m : on pose $A_m(x, y)$

$$\Rightarrow \begin{cases} x = e^{\frac{1}{m}-1} \\ y = -m e^{\frac{1}{m}-1} \end{cases} \text{ d'où } \ln x = \frac{1}{m} - 1 \Rightarrow \frac{1}{m} = 1 + \ln x$$

$$\Rightarrow m = \frac{1}{1 + \ln x} \text{ d'où } y = \frac{-x}{1 + \ln x} \quad / x > 0 \text{ et } x \neq e^{-1}$$

$$\Rightarrow A_m \text{ varie sur la courbe } E : y = \frac{-x}{1 + \ln x} \quad / x > 0 \text{ et } x \neq e^{-1}$$

$$2) \begin{cases} g(x) = \frac{-x}{1 + \ln x} \text{ si } x > 0 \text{ et } x \neq e^{-1} \\ g(0) = 0 \end{cases}$$

$$a) \subseteq \text{ de } g \text{ en } 0^+ : \lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} \frac{-x}{1 + \ln x}$$

$$= \lim_{x \rightarrow 0^+} \frac{-x}{1 + \ln x} = \frac{0}{-\infty} = 0 = g(0)$$

$$\Rightarrow g \subseteq \text{ en } 0^+$$

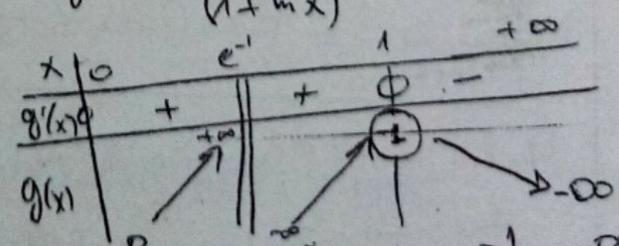
$$\cdot \subseteq \text{ de } g \text{ en } 0^+ : \lim_{x \rightarrow 0^+} \frac{g(x) - g(0)}{x - 0} =$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{-x}{1 + \ln x}}{x} = \lim_{x \rightarrow 0^+} \frac{-1}{1 + \ln x} = 0$$

$$\Rightarrow g \subseteq \text{ en } 0 \text{ et } g'(0) = 0$$

$$b) \forall x > 0 \text{ et } x \neq e^{-1}, g'(x) = \frac{-(1 + \ln x) + x(\frac{1}{x})}{(1 + \ln x)^2}$$

$$\Rightarrow g'(x) = \frac{-\ln x}{(1 + \ln x)^2}$$



$$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} \frac{-x}{1 + \ln x} = \lim_{x \rightarrow +\infty} \frac{-1}{\frac{1}{x} + \ln x} = -\infty$$

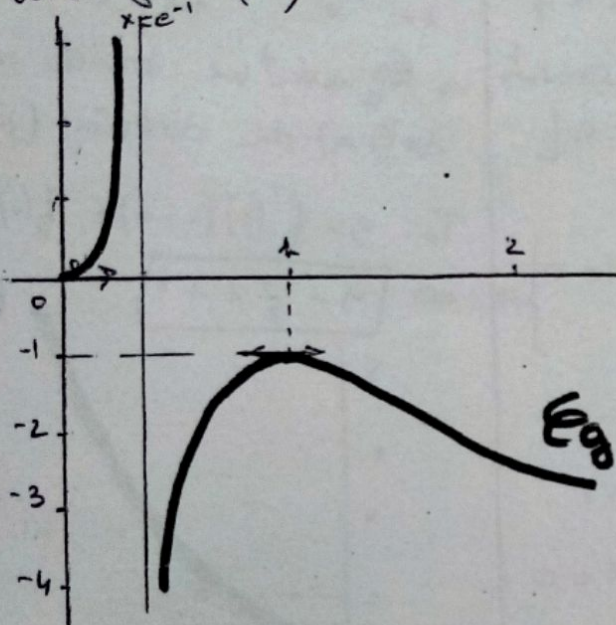
$$\begin{aligned} \lim_{(e^-)^-} g(x) &= \lim_{(e^-)^-} \frac{-x}{(e^-)^- (1+\ln x)} = +\infty \\ \lim_{(e^-)^+} g(x) &= \lim_{(e^-)^+} \frac{-x}{(e^-)^+ (1+\ln x)} = -\infty \end{aligned} \Rightarrow$$

$x = e^-$ Asymptote verticale

$$f(1) = \frac{-1}{1+0} = -1$$

$$\text{BI: } \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{-1}{1+\ln x} = 0$$

$\Rightarrow$ E_g admet une branche infinie au voisinage de $(+\infty)$ de direction $(0, \vec{i})$.



(B) $f(x) = f_1(x) = x \ln x - x$ si $x > 0$
 $f(0) = 0$

1) a) d'après (A) 1) $f \in \text{en ot } \text{non en ot}$

$$f'(x) = \ln x + x \left(\frac{1}{x} \right) - 1 = \ln x$$

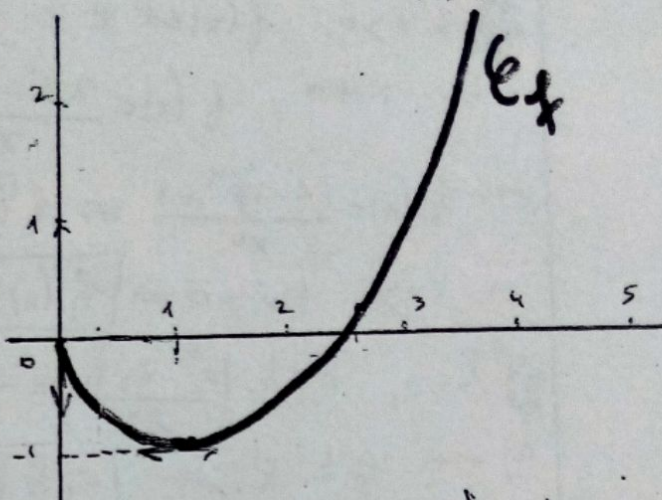
x	0	1	$+\infty$
$f'(x)$		- 0 +	
$f(x)$	0		$+\infty$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} (x \ln x - x) = \lim_{x \rightarrow +\infty} x(\ln x - 1) = +\infty \quad (\text{BI})$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} (\ln x - 1) = +\infty \Rightarrow E_g \text{ admet}$$

une BI en $(+\infty)$ de direction $(0, \vec{j})$

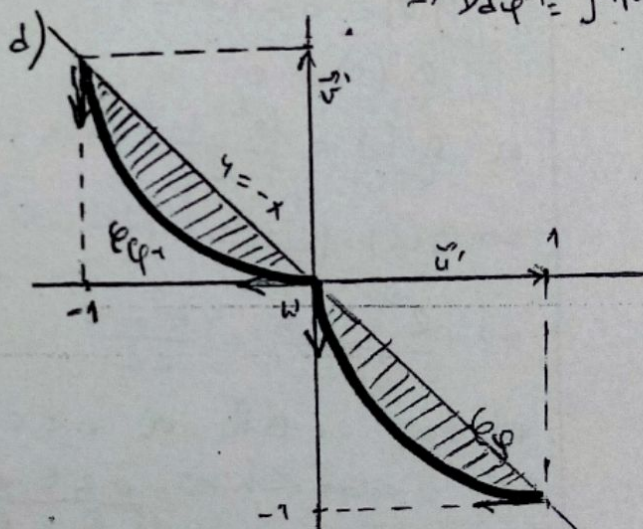
$$\begin{aligned} E_g \cap (0, \vec{i}) &\Leftrightarrow f(x) = 0 \Leftrightarrow x \ln x - x = 0 \\ &\Leftrightarrow x(\ln x - 1) = 0 \quad \begin{cases} x = 0 \\ \ln x = 1 \Leftrightarrow x = e \end{cases} \end{aligned}$$



$$\varphi(x) = f(x) \text{ si } x \in [0, 1]$$

$f \in \text{strat } \forall m [0, 1] \Rightarrow \varphi$ réalise une bijection de $[0, 1]$ sur $[-1, 0]$

c) $\varphi \in m]0, 1[\Rightarrow \varphi' \in m]-1, 0[$
 $\varphi'(x) \neq 0 \Rightarrow \varphi' \in \text{en } 0 \Rightarrow \text{Dap} \varphi =]-1, 0]$



$$2) a) I(a) = \int_a^1 (-x - \varphi(x)) dx = \int_a^1 (x - x \ln x + x) dx$$

$$\Rightarrow I(a) = \int_a^1 -x \ln x dx$$

$$\text{on pose } u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}$$

$$v(x) = -x \Rightarrow v'(x) = -\frac{1}{2} x^2$$

$$\Rightarrow I(a) = \left[-\frac{1}{2} x^2 \ln x \right]_a^1 + \frac{1}{2} \int_a^1 x dx$$

$$\Rightarrow I(a) = \left[-\frac{1}{2} x^2 \ln x + \frac{1}{4} x^2 \right]_a^1 = \frac{1}{4} + \frac{1}{2} a^2 \ln a - \frac{1}{4} a^2$$

$$\Rightarrow I(a) = \frac{1}{4} + \frac{a^2}{4} (2 \ln a - 1)$$

$$b) S = 2 \lim_{a \rightarrow 0^+} \int_a^1 (y - \varphi(x)) dx$$

$$\Leftrightarrow S = 2 \lim_{a \rightarrow 0^+} \int_a^1 (-x - \varphi(x)) dx$$

$$\Leftrightarrow S = 2 \lim_{a \rightarrow 0^+} \left[-\frac{x^2}{2} + \frac{e^2}{4} (2\ln a - 1) \right]$$

$$\Leftrightarrow S = \lim_{a \rightarrow 0^+} \left(\frac{1}{2} + \frac{a^2 \ln a}{2} - \frac{1}{2} a^2 \right) = \frac{1}{2}$$

$$3) U_0 = e \quad U_n > 0$$

$$U_{n-1} \cdot f'(U_n) = f(U_{n-1})$$

$$a) U_{n-1} \cdot \ln(U_n) = U_{n-1} \ln(U_n) + U_{n-1}$$

$$\Leftrightarrow U_{n-1} \cdot [\ln(U_n) - \ln(U_{n-1}) + 1] = 0$$

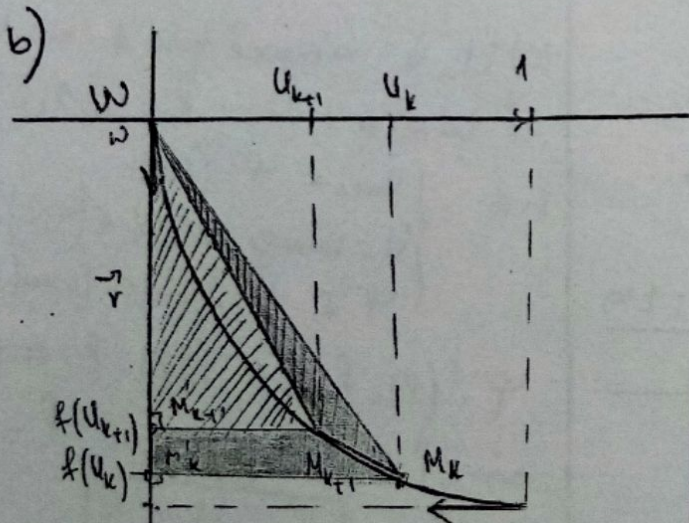
$$\text{or } U_{n-1} \neq 0 \Rightarrow \ln(U_n) - \ln(U_{n-1}) = -1$$

$$\Leftrightarrow \ln\left(\frac{U_n}{U_{n-1}}\right) = -1 \Rightarrow U_n = e^{-1} \cdot U_{n-1}$$

$\Rightarrow U_n$ est une suite géométrique de raison $q = e^{-1}$ et de 1^{er} ter $U_0 = e$

$$\Rightarrow U_n = U_0 \cdot q^n = e \cdot (e^{-1})^n = e^{1-n}$$

oncl: $\forall n \in \mathbb{N} \quad \boxed{U_n = e^{1-n}}$



Soit M'_k et M'_{k+1} : les proj orth de M_k et M_{k+1} sur l'axe des ordonnées

$\Rightarrow S_k$: l'aire du triangle $WM_k M'_{k+1}$

S_1 : l'aire du triangle $WM_k M'_k$

S_2 : l'aire du triangle $WM_{k+1} M'_{k+1}$
et S_3 : l'aire du trapèze $M_k M_{k+1} M'_{k+1} M'_k$

$$\Rightarrow S_n = S_1 - (S_2 + S_3)$$

$$\text{or } S_1 = \frac{1}{2} U_k \cdot |f(U_k)| = -\frac{1}{2} U_k \cdot f(U_k)$$

$$\text{et } S_2 = \frac{1}{2} U_{k+1} \cdot |f(U_{k+1})| = -\frac{1}{2} U_{k+1} \cdot f(U_{k+1})$$

$$\text{et } S_3 = \frac{|f(U_{k+1}) - f(U_k)| \cdot (U_k + U_{k+1})}{2}$$

$$\Rightarrow S_k = -\frac{1}{2} U_k f(U_k) + \frac{1}{2} U_{k+1} f(U_{k+1}) -$$

$$\left[\frac{1}{2} U_k f(U_{k+1}) + \frac{1}{2} U_{k+1} f(U_k) - \frac{1}{2} U_k f(U_k) - \frac{1}{2} U_{k+1} f(U_{k+1}) \right]$$

$$\Rightarrow S_k = \frac{1}{2} U_{k+1} f(U_k) - \frac{1}{2} U_k f(U_{k+1})$$

$$\Rightarrow S_k = \frac{1}{2} [U_{k+1} \cdot f(U_k) - U_k \cdot f(U_{k+1})]$$

$$\text{or } U_k = e^{1-k} \text{ et } f(U_k) = e^{1-k} (1-k) - e^{1-k} \Leftrightarrow f(U_k) = -k e^{1-k}$$

$$\Rightarrow S_k = \frac{1}{2} [e^{-k} \cdot (-k e^{1-k}) - e^{1-k} \cdot (-k-1) e^{-k}]$$

$$= \frac{1}{2} [-k e^{1-2k} + k e^{1-2k} + e^{1-2k}]$$

$$\Rightarrow \boxed{S_k = \frac{1}{2} e^{1-2k}}$$

$$c) S_n = \sum_{k=0}^{n-1} S_k = \sum_{k=0}^{n-1} \frac{1}{2} e \cdot (e^{-2})^k$$

$$\Leftrightarrow S_n = \frac{1}{2} e \left[\frac{1 - (e^{-2})^n}{1 - e^{-2}} \right]$$

$$\Leftrightarrow S_n = \frac{1}{2} e \left[\frac{1 - e^{-2n}}{1 - e^{-2}} \right] \cdot \frac{e^2}{e^2}$$

$$\Leftrightarrow S_n = \frac{1}{2} \frac{e^3}{e^2 - 1} (1 - e^{-2n})$$

$$\lim_{n \rightarrow +\infty} e^{-2n} = 0 \Rightarrow$$

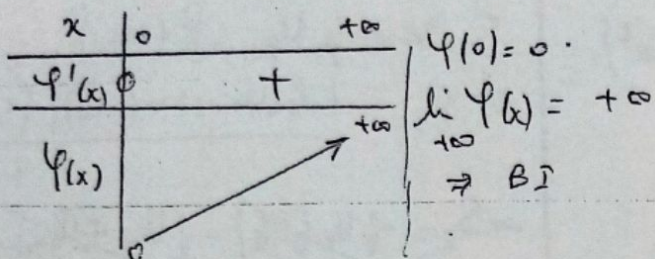
$$\boxed{\lim_{n \rightarrow +\infty} S_n = \frac{e^3}{2(e^2 - 1)}}$$

Problème 24 : BAC 1992 - Session de Contrôle

1) $\varphi(x) = (x-1)e^x + 1 \quad D\varphi = \mathbb{R}_+$

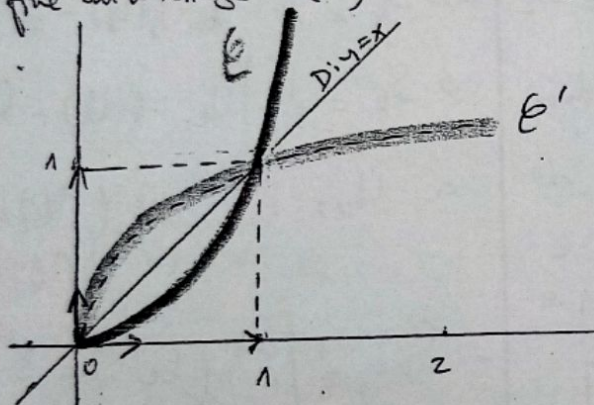
1a) $\varphi \in \mathcal{C}^1 \text{ sur } \mathbb{R}_+$

$\varphi'(x) = e^x + (x-1)e^x = xe^x \geq 0 \text{ sur } \mathbb{R}_+$



$\lim_{x \rightarrow +\infty} \frac{\varphi(x)}{x} = \lim_{x \rightarrow +\infty} \frac{(x-1)e^x + 1}{x} = \lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right) e^x = +\infty$

$= +\infty \Rightarrow E$ admet une branche infini au voisinage de $(+\infty)$ de direction $(0, \vec{j})$



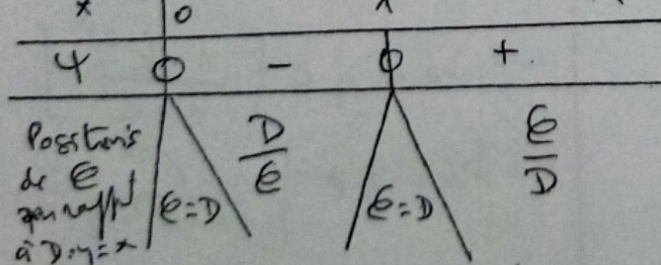
b) on pose $\psi(x) = \varphi(x) - x \quad D\psi = \mathbb{R}_+$

$\psi(x) = (x-1)e^x - (x-1)$

$\psi(x) = (x-1)(e^x - 1)$

or $\forall x \geq 0, e^x \geq 1 \Rightarrow e^x - 1 \geq 0$

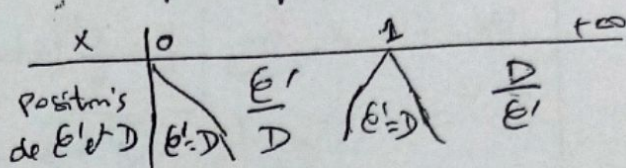
d'où $\text{sgn}(\psi'(x)) = \text{sgn}(x-1)$



2) a) $\varphi \in \mathcal{C}^1 \text{ sur } \mathbb{R}_+ \Rightarrow \varphi$ admet une bijection de $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ alors φ est une bijection sur $\mathbb{R}_+$

b) $E' = S_D(E) / D: y=x$

Comme $E' = S_D(E) \Rightarrow$ la position de E' par rapport à D sont inverses



3) $U_0 = 2, \quad U_{n+1} = \varphi^{-1}(U_n)$

a) $\forall n \in \mathbb{N} \quad U_n > 1$

pu $n=0, U_0 = 2 > 1$ vérifie

Supposons que $U_n > 1$ vrai $\rightarrow n$

or $\varphi \uparrow \Rightarrow \varphi^{-1} \uparrow \Rightarrow \varphi^{-1}(U_n) > \varphi^{-1}(1)$

$\Rightarrow U_{n+1} > 1$ car $\varphi(1) = 1 \Leftrightarrow \varphi^{-1}(1) = 1$

concl: $\forall n \in \mathbb{N} \quad \boxed{U_n > 1}$

b) $\forall n \quad U_n \searrow$

ma $\forall x \in]1, +\infty[; E'$ est située au dessous de $D \Leftrightarrow \varphi^{-1}(x) < x$

comme $U_n \in]1, +\infty[\Rightarrow \varphi^{-1}(U_n) < U_n$

$\Rightarrow U_{n+1} - U_n < 0 \Rightarrow \boxed{U_n \text{ strict } \searrow}$

c) $U_n \searrow$ minorée par 1 $\Rightarrow U_n$ converge

$\uparrow$ converge vers $l \in [1, 2]$

ma $\begin{cases} U_{n+1} = \varphi^{-1}(U_n) \\ U_n \text{ converge vers } l \in [1, 2] \\ \varphi^{-1} \in \text{sur } [1, 2] \subset]0, +\infty[\end{cases} \Rightarrow$

$\varphi^{-1}(l) = l \Leftrightarrow \varphi(l) = l \Leftrightarrow \varphi(l) = 0$

$\Rightarrow l = 0 \notin [1, 2]$ ou $l = 1$

concl: $\boxed{\lim U_n = 1}$

II

$\begin{cases} f(x) = \frac{e^x - 1}{x} & \text{si } x > 0 \\ f(0) = 1 \end{cases}$

1) si $x > 0, f(x) = \frac{e^x - 1}{x} \in \text{sur }]0, +\infty[$

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x} = 1 = f(0) \Rightarrow$

f est continue en 0^+
d'où f est continue sur $[0, +\infty[$

2) si $x > 0$, $f(x)$ est $\leq$ sur $]0, +\infty[$
Car $x \neq 0$, $f'(x) = \frac{xe^x - (e^x - 1)}{x^2}$

$$\Leftrightarrow f'(x) = \frac{(x-1)e^x + 1}{x^2} \Leftrightarrow f'(x) = \frac{\varphi(x)}{x^2}$$

$$\text{or } \forall x > 0, \varphi(x) > 0 \Rightarrow \boxed{f'(x) > 0} \forall x > 0$$

3) $h > 0$, $g(x) = \left(\frac{e^h - 1 - h}{h^2}\right)x^2 - e^x + x + 1$

a) on a $g \leq 0$ sur $[0, h] \subset \mathbb{R}_+$

$\Rightarrow$ d'après le théorème des accroissements finis, il existe au moins $c \in]0, h[$

tel que $g'(c) = \frac{g(h) - g(0)}{h - 0}$

$$\text{or } g(h) = \left(\frac{e^h - 1 - h}{h^2}\right)h^2 - e^h + h + 1 = 0$$

$$\text{et } g(0) = 0 - 1 + 1 = 0$$

$$g'(c) = 0$$

$$\text{or } g'(x) = \left(\frac{e^h - 1 - h}{h^2}\right)2x - e^x + 1$$

$$\Rightarrow g'(c) = \left(\frac{e^h - 1 - h}{h^2}\right)2c - e^c + 1 = 0$$

$$\Leftrightarrow \frac{e^h - 1 - h}{h^2} = \frac{e^c - 1}{2c}$$

$$\text{on pose } c = \theta h \text{ or } 0 < c < h$$

$$\Leftrightarrow 0 < \theta h < h \Leftrightarrow 0 < \theta < 1$$

$$\Rightarrow \text{il existe } \theta \in]0, 1[\mid \frac{e^h - 1 - h}{h^2} = \frac{e^{\theta h} - 1}{2\theta h}$$

$$b) \lim_{h \rightarrow 0^+} \left(\frac{e^h - 1 - h}{h^2}\right) = \lim_{h \rightarrow 0^+} \frac{e^{\theta h} - 1}{2\theta h} = l$$

$$\text{on pose } t = \theta h \text{ si } h \rightarrow 0^+ \Rightarrow t \rightarrow 0^+$$

$$\Rightarrow l = \lim_{t \rightarrow 0^+} \frac{e^t - 1}{2t} = \frac{1}{2}$$

$$\text{Ainsi : } \boxed{\lim_{h \rightarrow 0^+} \left(\frac{e^h - 1 - h}{h^2}\right) = \frac{1}{2}}$$

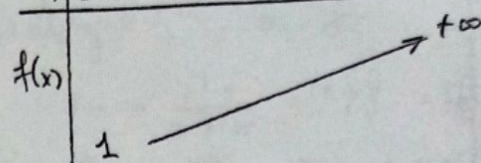
$$c) \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{\frac{e^x - 1 - x}{x^2} - 1}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$$

$\Rightarrow f$ est dérivable en 0^+ et $f'(0) = \frac{1}{2}$

4)

x	0	$+\infty$
$f'(x)$	$\frac{1}{2}$	$+$



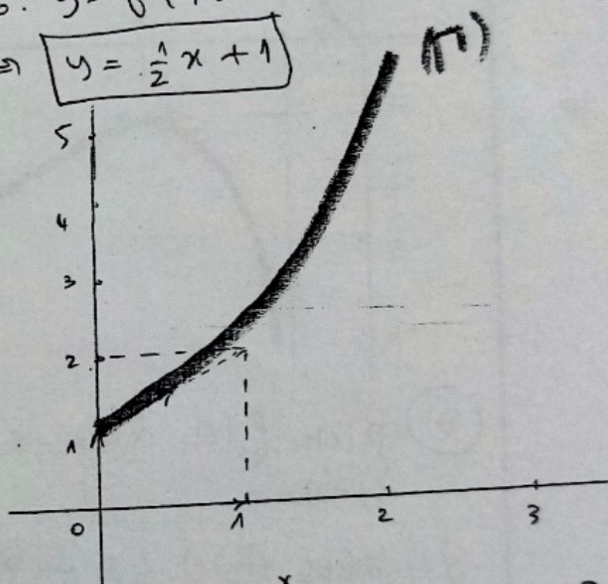
$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \left(\frac{e^x - 1}{x}\right) = \lim_{x \rightarrow +\infty} \frac{e^x}{x} = +\infty$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \left(\frac{e^x - 1}{x^2}\right) = \lim_{x \rightarrow +\infty} \frac{e^x}{x^2} = +\infty$$

$\Rightarrow f$ admet une branche inférieure au voisinage de $(+\infty)$ de direction $(0, \frac{1}{2})$.

To: $y = f'(0)(x - 0) + f(0)$ or $\left| \begin{matrix} f'(0) = \frac{1}{2} \\ f(0) = 1 \end{matrix} \right.$

$$\Rightarrow \boxed{y = \frac{1}{2}x + 1}$$



III $F(x) = \int_0^x f(t) dt$ $DF = \mathbb{R}$

1) a) on a $f(t)$ continue sur $\mathbb{R}_+$
et $\forall x \in \mathbb{R}, e^x > 0 \Rightarrow f \leq m(0, e^x)$
d'où F existe sur $\mathbb{R}$

b) f étant $\leq$ sur $\mathbb{R}_+ \Rightarrow f$ admet au moins une primitive H d $\mathbb{R}_+$
et telle que $H'(t) = f(t)$

$$\Rightarrow F(x) = [H(t)]_0^{e^x} = H(e^x) - H(0)$$

$$\text{or } x \mapsto e^x \leq \text{sur } \mathbb{R}$$

$\Rightarrow f(x)$ est dérivable sur $\mathbb{R}$

$$b \quad f'(x) = e^x - 1$$

$$\forall x \in \mathbb{R}, e^x > 0 \Rightarrow e^x > 1$$
$$\Rightarrow \boxed{f'(x) > 0} \quad \forall x \in \mathbb{R}$$

$$\Rightarrow F(x) \longrightarrow \int_0^x f(t) dt = 0$$

Concl : $\lim_{x \rightarrow -\infty} f(x) = 0$

3) a) on pose $u(t) = e^t - t$ $Du = \mathbb{R}^+$

$$u \equiv d \pmod{12} +$$

$$u'(t) = e^t - 1 \geq 0 \quad \forall t \geq 0$$

$t \mid 0 \quad t \rightarrow \infty$
 $u' \mid 0 \quad +$
 $u \mid 1 \quad \nearrow t \rightarrow \infty$

$U(0) = 1$

$$\rightarrow \forall t \geq 0 \quad u(t) \geq 1 \Rightarrow u(t) > 0$$

also $e^t \geq t \Leftrightarrow e^t - 1 \geq t - 1$

don't go, $\frac{e^t - 1}{t} \geq \frac{t-1}{t}$

Conce $e^x \geq 1 \Rightarrow \int_1^{e^x} \frac{t-1}{t} dt \geq \int_1^{e^x} \frac{t-1}{t} dt$

$$\Rightarrow \int_0^1 f(t) dt + \int_1^{e^x} b(t) dt \geq \int_1^{e^x} \frac{t-1}{t} dt + \int_0^1 g(t) dt$$

$$\Rightarrow \int_0^e f(t) dt \geq \int_1^e \frac{t-1}{t} dt + f(0)$$

$$\Rightarrow f(x) \geq f(0) + \int_0^x \frac{t-1}{t} dt$$

a $f(t) \geq 0 \Rightarrow 0 < e^x \Rightarrow f(x) > 0$

donc $F(t) > 0 \Rightarrow F(t) + \int_1^t \frac{t-1}{t} dt \geq \int_1^t \frac{t-1}{t} dt$

also $F(x) \geq \int_1^x \frac{t-1}{t} dt \quad \forall x \geq 0$

$$b) \int_1^{e^x} \frac{t-1}{t} dk = \int_1^{e^x} \left(1 - \frac{1}{t}\right) dk = \left[t - \ln t\right]_1^{e^x} \\ = e^x - x - 1 \Rightarrow \int_1^{e^x} \frac{t-1}{t} dk = e^x - x - 1$$

donc $F(x) \geq e^x - x - 1$

donc $f(x) \equiv e$

• $\lim_{x \rightarrow +\infty} (e^x - x - 1) = \lim_{x \rightarrow +\infty} x \left[\frac{e^x}{x} - 1 - \frac{1}{x} \right] = +\infty$

$$\Rightarrow \boxed{\lim_{x \rightarrow +\infty} f(x) = +\infty}$$

$$\text{für } x > 0 \Rightarrow \frac{f(x)}{x} \geq \frac{e^x - x - 1}{x}$$

$$\Leftrightarrow \frac{f(x)}{x} \geq \frac{e^x - 1 - x}{x} \quad | \cdot x \Rightarrow$$

$$\lim_{x \rightarrow \infty} \left(\frac{e^x}{x} - 1 - \frac{1}{6x} \right) = +\infty$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = +\infty$$

alors la courbe C_F admet une
branche infinie au voisinage de
($+\infty$) de direction $(\vec{OF})$.

4) $F(t) = \int_0^1 f(t) dt$: l'aire limitée par la courbe Ef , l'axe des abscisses et les droites $x=0$ et $x=1$

on divise $[0, 1]$ en 5 intervalles
d'amplitude $0,2 = \frac{1}{5}$

Problème 26 : BAC 1991 - Session de Contrôle

$$(A) \begin{cases} f_0(x) = \sqrt{x(2-x)} \\ f_n(x) = x^n \sqrt{x(2-x)} \end{cases} \quad D_n = [0, 2]$$

$$1) \cdot f_n \in \text{fon } x(2-x) \geq 0 \\ \Rightarrow f_n \in \text{fon } [0, 2] \quad \forall n \in \mathbb{N}$$

$$\cdot f_n \in \text{fon } x(2-x) > 0 \\ \Rightarrow f_n \in \text{fon }]0, 2[\\ \text{étudions alors la } \leq \text{ de } f_n \text{ en } 2^- \text{ et en } 0^+.$$

$$\cdot \text{ en } 2^-: f_n(2) = 0 \\ \lim_{x \rightarrow 2^-} \frac{f_n(x) - f_n(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{x^n \sqrt{x(2-x)}}{x - 2} \\ = \lim_{x \rightarrow 2^-} \frac{x^{n+1} (2-x) (-1)}{\sqrt{x} \cdot \sqrt{2-x} (x-2)} = \frac{-2^{n+1}}{0^+} = -\infty$$

$$\Rightarrow f_n \text{ n'admet pas } \leq \text{ en } 2^- \quad \forall n \in \mathbb{N} \\ \text{et } E_n \text{ admet en } (2, 0) \text{ une demi-tangente verticale dirigée vers le haut.}$$

$$\cdot \text{ en } 0^+: f_n(0) = 0 \\ \lim_{x \rightarrow 0^+} \frac{f_n(x) - f_n(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x^n \sqrt{x(2-x)}}{x} \\ = \lim_{x \rightarrow 0^+} x^{n-1} \sqrt{x(2-x)} = 0 \quad \text{si } n \geq 1 \\ = +\infty \quad \text{si } n = 0$$

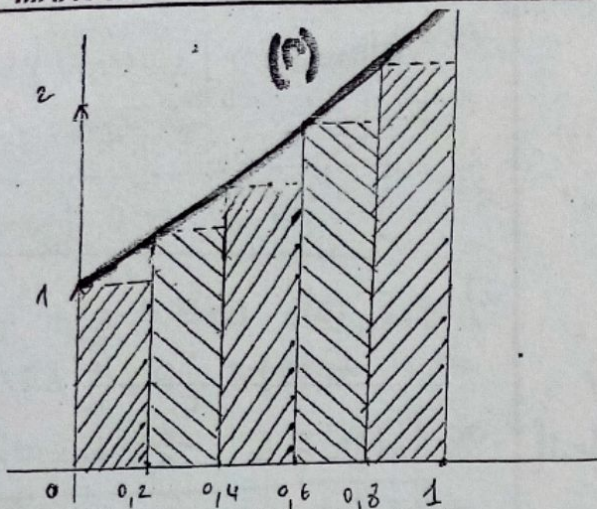
$$\Rightarrow \text{si } n = 0, f_n \in \text{fon }]0, 2[\\ \text{si } n \geq 1, f_n \in \text{fon } [0, 2]$$

$$2) \text{ Points fixes: } A(x, y) \in E_n \text{ then}$$

$$\Leftrightarrow \begin{cases} \text{ou } x=1 \longrightarrow y=1 \\ \text{ou } x=0 \longrightarrow y=0 \\ \text{ou } \sqrt{x(2-x)}=0 < \begin{cases} x=0 \longrightarrow y=0 \\ x=2 \longrightarrow y=0 \end{cases} \end{cases}$$

$$\text{Car: } \{0(0,0), A(1,1), B(2,0)\} \subset E_n$$

$$3) \text{ Si } n=0, f'_0(x) = (\sqrt{2x-x^2})' = \frac{2-2x}{2\sqrt{x(2-x)}} \\ \Rightarrow \forall x \in]0, 2[, f'_0(x) = \frac{1-x}{\sqrt{x(2-x)}}$$



$$A = \frac{1}{5} \left[\left(\frac{f(0)+f(0,2)}{2} \right) + \left(\frac{f(0,2)+f(0,4)}{2} \right) + \left(\frac{f(0,4)+f(0,6)}{2} \right) + \left(\frac{f(0,6)+f(0,8)}{2} \right) + \left(\frac{f(0,8)+f(1)}{2} \right) \right]$$

$$\Rightarrow A = \frac{1}{10} [f(0) + 2f(0,2) + 2f(0,4) + 2f(0,6) + 2f(0,8) + f(1)]$$

$$\text{or } f(0)=1, f(0,2) \approx 1,107, f(0,4) \approx 1,229 \\ f(0,6) \approx 1,37, f(0,8) \approx 1,532, f(1) \approx 1,778$$

$$\Rightarrow A \approx 1,32 \quad \text{Car: } [F(0) \approx 1,32]$$

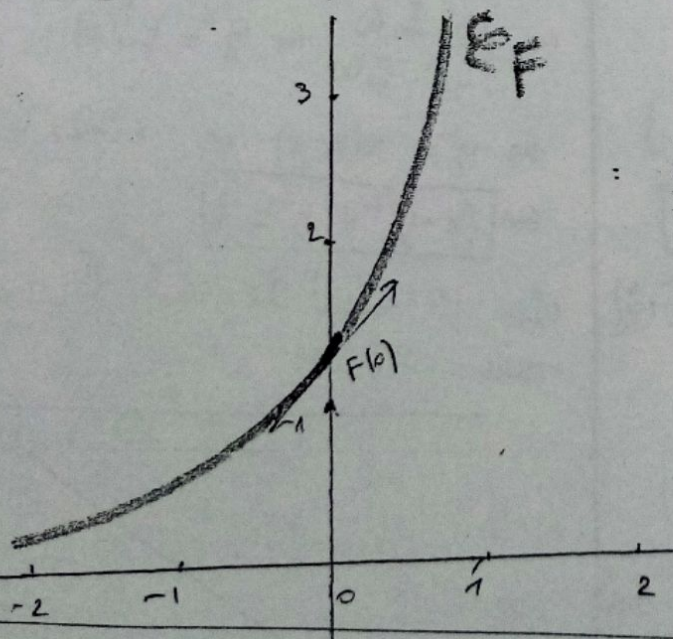
$$5) D_f = \mathbb{R}$$

$$F'(x) > 0$$

x	$-\infty$	$+\infty$
F'(x)		+
F(x)		$\nearrow +\infty$

• $y=0$ Asym en $(-\infty)$

• E_f admet une BI en $(+\infty)$ de direction $(0, f')$.



$$T_0: y = F'(0)(x-0) + F(0) \text{ avec } F'(0) = e^{-1} \approx 1,7$$

x	0	1	2
$f'_n(x)$		+	-
$f_n(x)$	0	↗ 1 ↘	0

• si $n \in \mathbb{N}^*$, $f'_n(x) = (x^n \sqrt{2x-x^2})'$

$$\Rightarrow f'_n(x) = n x^{n-1} \sqrt{2x-x^2} + x^n \frac{2-2x}{2\sqrt{2x-x^2}}$$

$$\Rightarrow f'_n(x) = \frac{n x^{n-1} (x(2-x) + x^n (1-x))}{\sqrt{2x-x^2}}$$

$$\Rightarrow f'_n(x) = \frac{x^n (2n - nx + 1 - x)}{\sqrt{x(2-x)}}$$

$$\Rightarrow f'_n(x) = \frac{x^n [(2n+1) - (n+1)x]}{\sqrt{x(2-x)}} \quad \forall n \in \mathbb{N}^*$$

on résout : $f'_n(x) = 0 \Leftrightarrow \begin{cases} x^n = 0 \rightarrow x=0 \\ \text{ou} \\ x = \frac{2n+1}{n+1} \end{cases}$

x	0	$\frac{2n+1}{n+1}$	2
$f'_n(x)$	0	+	-
$f_n(x)$	0	↗ $f_n(\frac{2n+1}{n+1})$ ↘	0

$$f_n\left(\frac{2n+1}{n+1}\right) = \left(\frac{2n+1}{n+1}\right)^n \sqrt{\frac{2n+1}{n+1} \left(\frac{1}{n+1}\right)} = \left(\frac{2n+1}{n+1}\right)^{\frac{n+1}{2}}$$

f'_n s'annule en $\left(\frac{2n+1}{n+1}\right) = u_n$ et change de signe de $+$ $\rightarrow$ $-$

$\Rightarrow f_n$ admet un maximum absolu en

$$u_n = \frac{2n+1}{n+1} \text{ égal à } v_n = f_n(u_n)$$

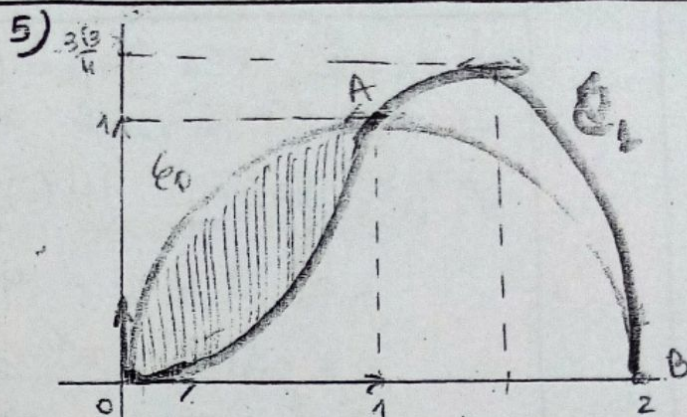
$$\text{avec } v_n = \left(\frac{2n+1}{n+1}\right)^n \frac{\sqrt{2n+1}}{n+1}$$

$$4) a) \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n}}{1 + \frac{1}{n}} = 2$$

$$\bullet \lim_{n \rightarrow \infty} f_n(u_n) = \lim_{n \rightarrow \infty} \ln[f_n(u_n)] = \lim_{n \rightarrow \infty} \ln[x] = -\infty$$

car $\lim_{n \rightarrow \infty} f(u_n) = f(2) = 0$

d'où $\lim_{n \rightarrow \infty} \ln(v_n) = -\infty$ d'où $\lim_{n \rightarrow \infty} v_n = 0^+$



⑥ $g(x) = \ln[f_0(x)] = \ln(\sqrt{x(2-x)})$

1) $D_g =]0, 2[$, $g'(x) = \frac{f'_0(x)}{f_0(x)}$

x	0	1	2
$g'(x)$		+	-
$g(x)$	$-\infty$	0	$-\infty$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} \ln(f_0(x)) = -\infty$$

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} \ln(f_0(x)) = -\infty$$

$$g(1) = \ln[f_0(1)] = \ln 1 = 0$$

2) $h(x) = g(x)$ si $x \in]0, 1[$

$g \in \text{stricte } \nearrow \text{ sur }]0, 1[\Rightarrow h \text{ réalise une bijection de }]0, 1[\text{ sur }]-\infty, 0[= g$

alors h admet une fonction réciproque h^{-1} définie par

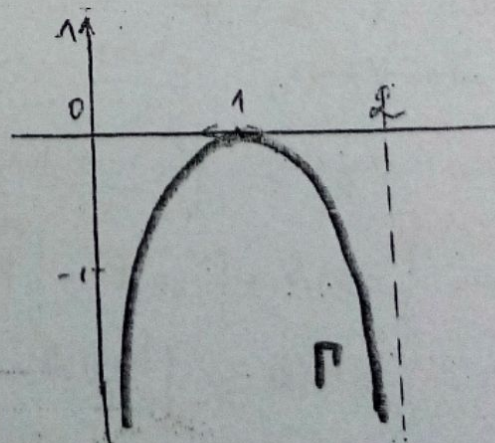
$$\begin{cases} \forall y \in]-\infty, 0[\\ h(y) = x \end{cases} \Leftrightarrow \begin{cases} \forall x \in]0, 1[\\ y = h^{-1}(x) \end{cases}$$

$$\bullet h(y) = x \Leftrightarrow \ln(\sqrt{y(2-y)}) = x \Leftrightarrow y(2-y) = e^x$$

$$\Rightarrow y^2 - 2y - e^x = 0, \quad \Delta' = 1 - e^x \geq 0$$

$$\text{donc } \begin{cases} y' = 1 - e^x \geq 0 \quad \forall x \in]-\infty, 0[\\ y'' = 1 + \sqrt{1 - e^x} \neq 0 \end{cases} \text{ et } [0, 1[\text{ à rejeter}$$

On a : $h^{-1}(x) = 1 - \sqrt{1 - e^x} \quad \forall x \leq 0$



$$3) a) \alpha \in]0, 1[$$

$$g(x) = g(\alpha) \Rightarrow \ln[f(x)] = \ln[f(\alpha)]$$

$$\Rightarrow f_0(x) = f_0(\alpha) \Rightarrow \sqrt{2x-x^2} = \sqrt{2\alpha-\alpha^2}$$

$$\Rightarrow 2-x^2 = 2\alpha-\alpha^2$$

$$\Rightarrow x^2 - 2x + (2\alpha - \alpha^2) = 0$$

$$\Delta' = 1 - (2\alpha - \alpha^2) = \alpha^2 - 2\alpha + 1 = (\alpha - 1)^2$$

$$\Rightarrow \sqrt{\Delta'} = |\alpha - 1| = 1 - \alpha \quad \text{car } \alpha \in]0, 1[$$

$$\Rightarrow x' = 1 - (1 - \alpha) = \alpha \in]0, 2[$$

$$x'' = 1 + (1 - \alpha) = 2 - \alpha \in]0, 2[$$

$$\Rightarrow S_{]0, 2[} = \{ \alpha, 2 - \alpha \}$$

$$b) G = h^{-1} \circ g \Rightarrow h^{-1}[g(x)] = G(x)$$

$$\text{or } \forall x \in]0, 2[, g(x) \in]-\infty, 0] = D_{f_1}$$

$$\Rightarrow DG =]0, 2[$$

$$\cdot \text{ si } x \in]0, 1[\quad , G(x) = h^{-1}(h(x)) = x$$

$$\cdot \text{ si } x \in [1, 2[\quad , G(x) = h^{-1}(g(x)) = 2 - x$$

$$c) f_n(\theta) = \int_0^{1+\sin\theta} f_n(t) dt \quad \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$1) a) \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], -1 \leq \sin\theta \leq 1$$

$$\Rightarrow 0 \leq 1 + \sin\theta \leq 2$$

$$\Rightarrow [0, 1 + \sin\theta] \subset [0, 2]$$

$$\text{comme } f_n \in \mathcal{M}([0, 2])$$

$$\Rightarrow f_n \text{ est intégrable sur } [0, 1 + \sin\theta]$$

$$\text{alors } f_n \text{ est définie sur } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$b) \text{ on a } \theta \mapsto 1 + \sin\theta \in \mathcal{M}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{et } (1 + \sin\theta) \in [0, 2] = D_{f_n}$$

$$\Rightarrow f_n \text{ est } \mathcal{M}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$F'_n(\theta) = (1 + \sin\theta)' \cdot f_n(1 + \sin\theta)$$

$$\Rightarrow F'_n(\theta) = \cos\theta \cdot (1 + \sin\theta)^n \sqrt{(1 + \sin\theta)(2 - \sin\theta)}$$

$$\Rightarrow F'_n(\theta) = \cos\theta (1 + \sin\theta)^n \sqrt{1 - \sin^2\theta}$$

$$\Rightarrow F'_n(\theta) = \cos\theta (1 + \sin\theta)^n |\cos\theta|$$

$$\text{or } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos\theta \geq 0$$

$$\text{donc } F'_n(\theta) = \cos^2\theta (1 + \sin\theta)^n$$

$$2) a) F'_0(\theta) = \cos^2\theta = \frac{1}{2} + \frac{1}{2} \cos(2\theta)$$

$$\Rightarrow f_0(\theta) = \frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) + k_0$$

$$\text{or } f_n\left(\frac{\pi}{2}\right) = 0 \Rightarrow -\frac{\pi}{4} + k_0 = 0 \Rightarrow k_0 = \frac{\pi}{4}$$

$$\Rightarrow F_0(\theta) = \frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) + \frac{\pi}{4}$$

$$\begin{aligned} F'_1(\theta) &= \cos^2\theta (1 + \sin\theta) \\ &= \cos^2\theta + \sin\theta \cdot (\cos\theta)^2 \end{aligned}$$

$$\Rightarrow F'_1(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta) - (\sin\theta)(\cos\theta)$$

$$\Rightarrow F_1(\theta) = \frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) - \frac{1}{3}(\cos\theta)^3 + k_1$$

$$\text{or } F_1\left(\frac{\pi}{4}\right) = 0 \Rightarrow \frac{\pi}{8} + k_1 = 0 \Rightarrow k_1 = -\frac{\pi}{8}$$

$$\text{donc } F_1(\theta) = \frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) - \frac{1}{3} \cos^3\theta + \frac{\pi}{8}$$

$$b) A = \int_0^1 (f_0(x) - f_1(x)) dx = \int_0^1 f_0(x) dx - \int_0^1 f_1(x) dx$$

$$\Rightarrow A = F_0(1) - F_1(1) = \frac{\pi}{4} - \left(\frac{1}{2} + \frac{\pi}{8}\right) \Rightarrow A = \frac{1}{8}$$

$$3) E'_0: y = -f_0(x) \quad , \quad E = E_0 \cup E'_0$$

$$M(x, y) \in E \Rightarrow \begin{cases} M \in E_0 \\ \text{ou} \\ M \in E'_0 \end{cases}$$

$$\Rightarrow \begin{cases} y = f_0(x) \\ \text{ou} \\ y = -f_0(x) \end{cases} \Rightarrow y^2 = f_0^2(x)$$

$$\Rightarrow y^2 = x(2-x) \Rightarrow x^2 - 2x + y^2 = 0$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$

$$\text{alors } M \text{ décrit le cercle } \mathcal{C}(1, 1)$$

$$\text{avec } \Omega(1, 0)$$

